<u>IMPORTANT</u>: Solutions must be submitted on <u>special blanks</u> – one problem per blank. Each blank should have <u>your name and problem number</u> in the appropriate text field – blanks without that information will be disregarded. Several blanks per problem can be used provided that each blank has your name and correct problem number. Problems 1-5 are compulsory for all applicants. Rest of the problems are program-specific – they are compulsory or give extra points for Master programs indicated in the title of the problem.

1. Find the derivative with respect to x of the functions

$$\text{(a) } x^{\cos x} \qquad \text{and} \qquad \text{(b) } \int_{\sin^2(x^2)}^{e^{2x}} \sin\left(\xi^2\right) d\xi$$

2. (a) Find the integral $\int_1^{e^2} \ln x \, dx$.
   (b) Expand the function $\left[\cos\left(x^3\right)\right]^{-1/2}$ into the Taylor series around $x = 0$ up to $o\left(x^6\right)$.

3. Solve the differential equation $xy' + 2y = 3x^2$, $y(1) = 1$.

4. Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

   (a) Find eigenvalues and eigenvectors of the matrix $\mathbf{A}$.
   (b) Find $\max\limits_{\mathbf{x}} \frac{|(\mathbf{Ax},\mathbf{x})|}{(\mathbf{x},\mathbf{x})}$, where $(\cdot,\cdot)$ is a dot product of vectors and the maximization is performed over all $\mathbf{x} = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3$, such that $\sum_{i=1}^3 x_i = 0$.

5. A father suggests two algorithms to divide a round pie between his sons: A) The elder son gets 2/3, and the younger son gets 1/3; B) The pie is cut along the line passing through two points chosen randomly at its circumference, and the younger son gets the smaller piece. Which algorithm gives a larger mathematical expectation of the younger son's part?

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6. **Compulsory problem for "Data Science" and "Computational Science and Engineering"**: Let X be a set of N points in $\mathbb{R}^2$. Suppose that these points have distinct coordinates in the sense that for any (x, y) and (x', y') from X we have $x \neq x'$ and $y \neq y'$. Let us call the point (x, y) from X *Pareto-optimal* if there exists no other point (x', y') from X such that $x' < x$ and $y' < y$. Prove that all the Pareto-optimal points in X can be found using $O(N \log N)$ comparisons of coordinates.

7. **Compulsory problem for "Mathematical Physics"**: Two material points of mass m are moving without a friction on two concentric circles in some plane with radii R and r $(r < R)$. The material points are connected by a weightless spring of stiffness k, which has a negligible length in the free state. Find the Lagrangian for this mechanical system, write the Euler-Lagrange equation and find conserved quantities (integrals of motion), if any.

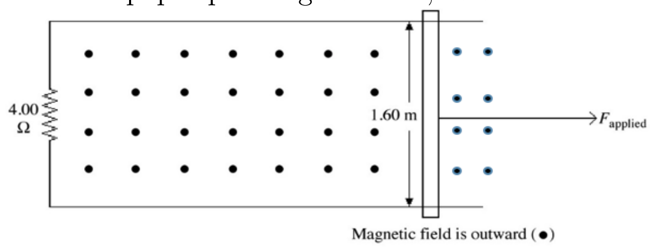
8. **Chemistry problem (extra points for "Materials Science")**: A portion of some organic compound X (weight 5.7 g) having no branching in the hydrocarbon skeleton has been combusted in oxygen, which produced carbon dioxide and water as the only products. Note that a full combustion of 1 mole of X requires 7.5 moles of oxygen. All combustion products were absorbed completely by 20% aqueous potassium hydroxide. The obtained solution was freeze-dried yielding a solid with a constant weight of 37.6 g. This solid was annealed at 200°C under normal pressure, which resulted in the weight loss of 3.1 g. It is also known that compound X reacts with aqueous KOH and can be hydrolyzed under acidic conditions. Acid-induced hydrolysis of X produces two compounds A and B, which have the same empiric formula, but different molecular weights.
   (a) Identify molecular formula of X.
   (b) Identify structure of X, which satisfies all the aforementioned conditions.
   (c) Write equations of all described reactions and identify also structures of compounds A and B.

9. **Physics problem (compulsory for "Photonics and Quantum Materials", extra points for "Materials Science")**: A quantum particle of mass m is moving in a one-dimension potential $U(x)$ such that $U(x > 0) = \alpha x^2$ (α is positive) and $U(x < 0) = +\infty$.

   (a) Find the energy spectrum. *Hint*: Use the symmetry of wave functions for a harmonic oscillator.

   (b) The potential is disturbed by a quartic term: $U(x > 0) = \alpha x^2 - \beta x^4$ (α and β are positive), which renders all bound states metastable. Assuming $\beta^2 \ll m\alpha^3/\hbar^2$, estimate the decay rate of the former ground state.

10. **Physics problem (compulsory for "Photonics and Quantum Materials", extra points for "Materials Science")**: A conducting bar moves along frictionless conducting rails connected to a 4 Ω resistor as shown in the figure. The length of the bar is 1.6 m and a uniform magnetic field of 1.2 T is applied perpendicular to the paper pointing outward, as shown in the figure.



   (a) What is the applied force required to move the bar to the right with a constant speed of 6 m/s?

   (b) What is the power dissipated in the 4 Ω resistor?

11. **Compulsory problem for "Energy Systems"**: The dynamics of an overdamped particle positioned at $x \in \mathbb{R}$ immersed into thermal bath is modeled by the Langevin (stochastic differential) equation

$$\eta \frac{dx}{dt} = -\frac{d}{dx} V(x) + \xi(t),$$

   where η is a friction coefficient, $V(x)$ is a potential. $\xi(t)$, representing the effect of the thermal bath on the particle, is modeled with as a Gaussian white noise with zero mean, $\langle \xi(t) \rangle = 0$, satisfying the fluctuation-dissipation relation: $\langle \xi(t)\xi(s) \rangle = 2\eta T \delta(t - s)$, where T is the temperature of the bath.

   (a) Consider the case of a confined potential bounded from below, $V(x) \geq 0$, $\lim_{x\to\pm\infty} V(x) \to \infty$. Find stationary (time-independent) probability distribution for the particle to be at position x. Does the particle move in average, i.e. is the "current", $d\langle x \rangle/dt$, nonzero?

   (b) The same equation, when the potential is periodic in space with the spatial period L, $\forall x$: $V(x+L) = V(x)$, and the temperature is changing periodically in time with the temporal period τ, $\forall t :\ T(t+\tau) = T(t)$, represents the stochastic dynamics of a thermal ratchet. Can the average current be nonzero in this case? Argue physically and/or back it up mathematically, by choosing an exemplary potential and temperature profiles and resolving the Langevin equation.

12. **Compulsory problem for "Energy Systems"**: Consider an electric power network with m generators and n loads (consumers). Generator's production is limited by $0 \leq P_j \leq P_{max},\ j = 1, \ldots, m$, where P_{max} represents its capacity. Assume that the power transfer factors a_{jk}, that represents the power that flows from generator j to load k, and power consumed by load, $L_k, k = 1, \ldots, n$, are known.

    1. Propose a mathematical formulation aiming to achieve optimal generation dispatch meeting demand profile L_k, $k = 1, \ldots, n$. Hint: introduce a unit production cost as you find fit.
    2. Extend the formulation, additionally requiring that
       (a) No more than half of the total power $\sum_{k=1}^{n} L_k$ is provided by any $\ell \sim m/10$ generators;
       (b) No more than half generators are on ($P_j > 0$).
    3. Does adding constraint (a) or constraint (b), or both (a) and (b), leads to optimization problems that are significantly more complex to solve than the original formulation?