AN ADAPTIVE WAVELET METHOD FOR FLUID–STRUCTURE INTERACTION

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Abstract One of the most practically important problems in aerodynamics is calculating moderate to high Reynolds number flow around solid obstacles of arbitrary shape. This problem arises in aerodynamics (e.g. turbulent flow over the wings and fuselage of airplanes), in off-shore drilling (e.g. water flow around riser tubes transporting oil from the sea bed to the surface), and in the wind engineering of buildings. In each case the primary difficulty arises from the need to calculate turbulent or transitional flow with boundary conditions on complicated domains. In addition, it may be important to allow for the obstacle to move or deform in response to the applied fluid forces (this motion in turn affects the flow).

In this paper we propose combining two mathematical approaches to calculate fluid–structure interaction numerically. The first technique, the adaptive wavelet method, tackles the problem of efficiently resolving a high Reynolds number flow in complicated geometries (where grid resolution should depend both on time and location). The second technique, Brinkman penalization, addresses the problem of efficiently implementing solid boundaries of arbitrary complexity.

Keywords:
Turbulence, fluid-structure interaction, wavelets, penalization

Introduction

Adaptive wavelet methods have been developed recently to solve the Navier–Stokes equations at moderate Reynolds numbers (e.g. Schneider et al., 1997; Vasilyev and Bowman, 2000). The adaptive wavelet method is appropriate to turbulence since the wavelets (which are localized in both space and scale) adapt the numerical resolution naturally to the intermittent structure of turbulence at small scales . The wavelet method thus allows turbulent flows to be calculated with a greatly reduced number of modes with a well-controlled error. Furthermore, the computational cost of the algorithm is independent of the dimensionality of the problem and is $O(N)$, where N is the total number of wavelets actually used. We employ a collocation wavelet method using second generation wavelets, which allows the order of the method to be easily varied (we generally use an 8th-order method). Another advantage of the collocation wavelet approach is that it should be straightforward (in principle) to go from two dimensions (the case considered here) to three dimensions.

Parallel to the development of efficient wavelet codes for turbulence, we have been investigating the use of the Brinkman equation to simulate the presence of arbitrarily complex solid boundaries (Kevlahan and Ghidaglia, 2001). This technique allows boundary conditions to be enforced to a specified precision, without changing the numerical method (or grid) used to solve the equations. The main advantage of this method, compared to other penalization type methods, is that the error can be estimated rigorously in terms of the penalization parameter. It can also be shown that the solution of the penalized equations converges to the exact solution in the limit as the penalization parameter tends to zero. The disadvantage of this method is that it makes the equations stiff. In order to overcome this limitation we use a stiffly-stable Krylov subspace method in time (Edwards et al., 1994). This method is variable order and uses an adaptive step size.

In the following section we briefly outline the main features of the numerical method, with an emphasis on its differences compared with other commonly used simulation techniques. §2 presents some results that illustrate the capabilities of our approach.

1. Numerical method

1.1 Penalized vorticity equation

We resolve the two-dimensional vorticity equations, to which we add a volume Brinkman penalization term (see (Khadra et al., 2000) for more

information about Brinkman penalization in velocity-pressure equations). We choose the vorticity formulation, rather than primitive variables formulation, for three reasons. First, in two dimensions the vorticity formulation is simpler since vorticity is a pseudo-scalar. Secondly, vorticity is more compact than velocity, which allows for greater compression. Finally, vorticity is the physically important quantity responsible for fluid forces and vortex generation. By working directly with the vorticity we expect to calculate these quantities more accurately.

The penalized two-dimensional vorticity-velocity equations may be written in the following form

$$\frac{\partial \omega}{\partial t} = -\boldsymbol{u} \cdot \nabla \omega + \nu \Delta \omega - \frac{1}{\eta} \nabla \times [\chi(\boldsymbol{x}, \boldsymbol{x}_0, t)(\boldsymbol{u} - \boldsymbol{u}_O)], \tag{1}$$

where ω is the vorticity, $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{x}_0$ is the obstacle centre, $\boldsymbol{u}_0$ is the obstacle velocity, ν is the viscosity, and

$$\chi(\boldsymbol{x}, \boldsymbol{x}_o, t) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \text{solid}, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

is a mask function that defines the location of the solid obstacles.

The last term on the rhs of (1) is a penalization term that approximates the no-slip boundary conditions on the velocity as $\eta \to 0$ (Angot et al., 1999). Note that (1) is defined over the whole computational domain (both fluid and solid). The solid is now viewed as a porous medium with very small permeability $\propto 1/\eta$. In vorticity form the penalization term is responsible for eliminating vorticity from the interior of the solid and for generating vorticity at the fluid-solid interface. Angot et al., 1999 have shown mathematically that the error due to the penalization is $||\boldsymbol{u} - \boldsymbol{u}_\eta|| \leq C\eta^{1/4}$, although in practice we find the error is $O(\eta)$.

The penalization approach allows us to calculate flow around solid obstacles of arbitrary complexity (that may also be moving) simply by modifying the mask function χ in the appropriate way. A simple calculation shows that the resultant fluid force $\boldsymbol{F}$ on the obstacle may be found at little cost using the following volume integral,

$$\boldsymbol{F}(t) = \frac{1}{\eta} \int \chi(\boldsymbol{x}, \boldsymbol{x}_0, t)(\boldsymbol{u} - \boldsymbol{u}_0) \, \mathrm{d}\boldsymbol{A}, \tag{3}$$

where the integral is taken over the entire computational domain. This volume integral is easier to calculate and is often more accurate than the surface integrals usually employed to calculate the fluid force.

To allow for full fluid-structure interaction, the motion of the obstacle's centre of mass $\boldsymbol{x}_o(t)$ is modelled as a forced oscillator (Khalak and

Williamson, 1999),

$$m\ddot{\boldsymbol{x}}_0(t) + c\dot{\boldsymbol{x}}_0(t) + k\boldsymbol{x}_0(t) = \boldsymbol{F}(t), \tag{4}$$

where m is the obstacle's mass, c is the mechanical damping, k is the spring constant and $\boldsymbol{F}(t)$ is the fluid force (calculated using 3). The oscillator equation and vorticity equation are thus coupled via the force $\boldsymbol{F}$ and the mask χ.

Equations (1) and (4) form a closed set of equations describing two-dimensional fluid–structure interaction.

1.2 Adaptive wavelet numerical method

The penalized vorticity equations introduced in the previous section are solved using an collocation wavelet method in space (Vasilyev and Bowman, 2000) and an adaptive Krylov method in time (Edwards et al., 1994). Both methods are adaptive and of variable order. Note that the wavelet basis functions are localized in both position and scale, and are thus well-adapted to the multi-scale vortex structures of turbulence and fluid–structure interactions.

In the collocation wavelet method there is a one-to-one correspondence between grid points and wavelets, which makes calculation of nonlinear terms simple (compared with Galerkin wavelet methods) and allows the grid to adapt automatically and dynamically to the solution by adding or removing wavelets. Very briefly, at each time step we take the wavelet transform of ω, remove all wavelets with coefficient magnitude less than a threshold ϵ, and then reconstruct the solution. It can be shown that the L_∞ error of this approximation is bounded by ϵ. To account for the evolution of the solution over one time step we add the nearest neighbour wavelet coefficients in position and scale. Since each wavelet corresponds to a single grid point this procedure allows the grid to automatically follow the evolution of the solution in position and scale. The derivatives are calculated using finite differences on the refined grid. We use second generation wavelets (Sweldens, 1998), which allow the order of the wavelet (and hence of the finite differences) to be easily varied. In the results presented here we use fourth- or eighth-order derivatives. The wavelet method has a computational complexity $O(N)$, where N is the number of wavelets retained in the calculation (i.e. those wavelets with coefficients greater than ϵ plus nearest neighbours).

Because we are working with the vorticity equation we need a method of calculating the velocity from the vorticity. We have decided to use a sort of vortex method, where each wavelet is treated as a vortex with a given circulation (found from its coefficient and scale). The velocity is then found from these 'wavelet-vortices' using the Biot–Savart law. In

practice, we take advantage of the Fast Multipole Method (FMM) (Greengard and Rokhlin, 1987), which performs this calculation in $O(N)$ complexity, consistent with the rest of the numerical scheme.

Finally, we chose the Krylov scheme to integrate in time because it is stiffly-stable (for the linear part of the equation), the error can be controlled at each time step, and the order of the method can be easily varied by changing the dimension of the Krylov sub-space. A stiffly-stable method is necessary because of the small parameters ν and η.

The exterior boundary conditions may be either periodic, Dirichlet, or von Neumann. The results presented in §2 use Brinkman penalization to impose $\omega = 0$ on all four exterior boundaries of the rectangular computational domain. The exit (downstream boundary $x = x_{max}$) has a slow transition from fluid to solid to absorb vorticity without inducing unnecessary grid refinement.

In summary, we have developed an adaptive, variable order method for calculating fluid–structure interactions of arbitrarily complex geometry in two dimensions. Because the computational grid automatically adapts to the solution (in position and scale), we do not have to know *a priori* where the obstacle or vortices will be. In the following section we apply the method sketched here to some examples of fluid–structure interaction with fixed and moving obstacles.

2. Results

For simplicity (and so we can compare the results with other simulations and experiments) we will consider only cylindrical obstacles here. In all cases we impose a positive uniform mean flow $\boldsymbol{U} = (1, 0)$ in the horizontal x-direction about a cylinder with diameter $D = 1$. The Reynolds number of the flow is thus defined as $Re = |\boldsymbol{U}|D/\nu = 1/\nu$. We choose the following values for the penalization and wavelet refinement parameters respectively: $\eta = 10^{-3}$ and $\epsilon = 10^{-5}$.

We first consider flow around a fixed cylinder at a moderate Reynolds number. The computational domain has dimensions $[-2.5, 2.5]\times[-2.5, 2.5]$ and the maximum resolution is 256^2. Figure 1 shows the drag around an impulsively started cylinder as a function of time at $Re = 550$ calculated with the present method compared with the vortex method of (Koumoutsakos and Leonard, 1995). Note the good agreement at all times. This calculation is a good test of our method since drag is very sensitive to even slight errors in vorticity generation at the obstacle's surface.

We now consider the case of coupled fluid–structure interaction where the cylinder is free to move in the transverse y-direction. Because this

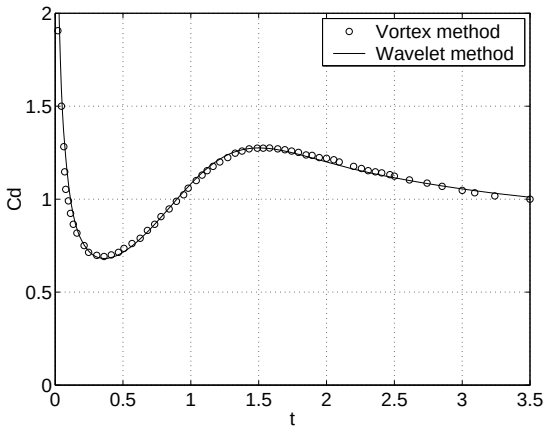

Figure 1. Drag around an impulsively started cylinder at $Re = 550$. Comparison with the vortex method (Koumoutsakos and Leonard, 1995).

flow is known to become three-dimensional for $Re > 180$, we ensure a physically two-dimensional flow by choosing $Re = 100$. The computational domain has dimensions $[-2.5, 17.5] \times [-10, 10]$ and the maximum resolution is 512^2. The mechanical frequency k is set equal to the Strouhal fluid frequency, $k = St = 0.167$ and added mass $m^* = m/(\pi/4) = 2$ (i.e. the mass of the cylinder is twice the mass of the fluid it displaces). These conditions ensure that the coupled fluid–mechanical system is close to resonance. Figure 2 shows the lift and centre of the cylinder as function of time. The cylinder undergoes large harmonic oscillations (with amplitude equal to the cylinder radius) in phase with the lift force. These results are similar to those observed in experiments (Khalak and Williamson, 1999).

Figure 3 shows how the wavelet method automatically adapts the grid in position and scale to follow the vortex gradients and the (changing) location of the cylinder. Note that grid is very fine in the boundary layer, but then coarsens downstream as the vortices diffuse. For this calculation we use only 8% of the maximum possible 512^2 modes.

3. Summary

In this paper we have presented an adaptive wavelet method for coupled fluid–structure interaction which uses a Brinkman penalization method to impose solid boundary conditions and a Krylov time scheme.

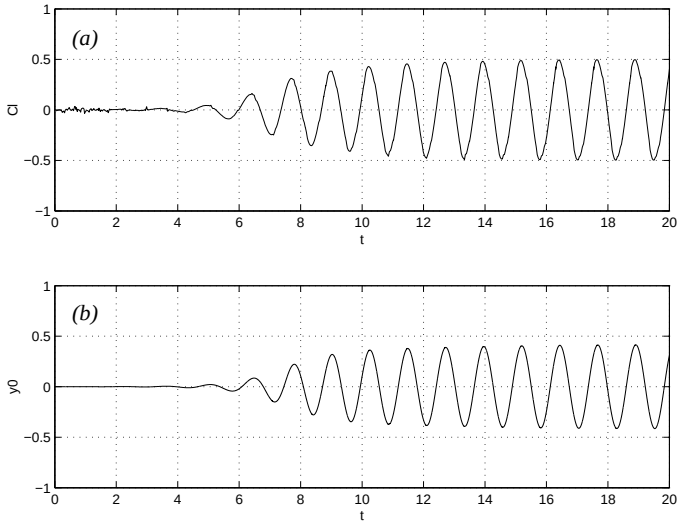

Figure 2. Fluid–structure interaction at Re = 100 (time normalized by mechanical frequency). (a) Lift. (b) y-position of cylinder centre.

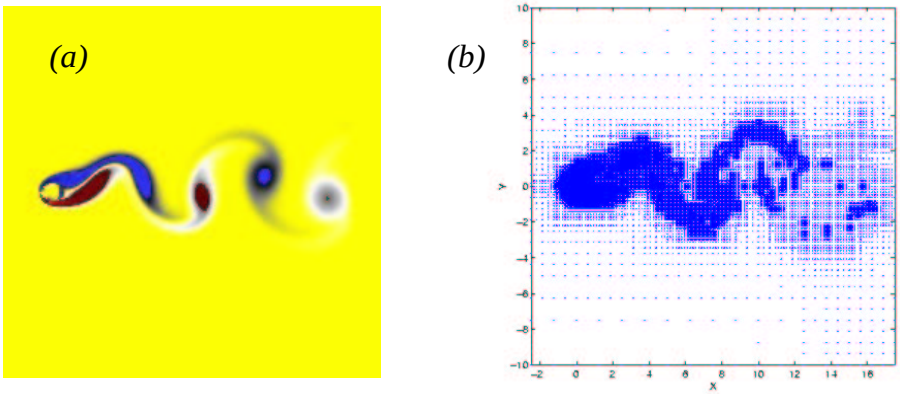

Figure 3. Fluid–Structure interaction at Re=100, t = 14. (a) Vorticity field. (b) Wavelet-adapted grid.

This approach gives accurate simulations with a greatly reduced number of modes compared with non-adaptive methods (about 16 times compression for the cases considered here). Because the grid refinement is automatic and dynamic, we do not need to know the location of the obstacle or vortices *a priori*. Our approach is therefore well-adapted to fluid-structure interaction problems where the obstacle moves or deforms due to the fluid forces.

In the future we intend to extend the method to three dimensions, and parallelize the algorithm. We will also consider more complicated obstacles (e.g. airfoils) and verify the method more extensively against experimental data.

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