A Characteristic-Based Volume Penalization Method for Arbitrary Mach Flows Around Solid Obstacles

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1 Introduction

Volume penalization is a subclass of immersed boundary methods for modeling complex geometry flows, which introduces the effects of obstacles by modifying the governing equations. The method presented in this paper encompasses general boundary conditions as an extension of the Brinkman Penalization Method (BPM) [1], which was originally developed for solid, isothermal obstacles in incompressible flows. A principal strength of Brinkman penalization is that error can be rigorously controlled *a priori*, with the solution converging to the exact in a predictable fashion [4, 5].

While much work has been done to refine BPM for various numerical techniques and flow regimes, boundary conditions have lacked generality, especially for compressible flows. They have been typically limited to slip and no-slip conditions for the inviscid and viscous flow around isothermal obstacles, though additional boundary conditions have been developed on a problem specific basis. In this way, BPM has been inapplicable and inflexible for many fluid problems, notably those demanding heat-flux and insulating boundary conditions on solid surfaces.

The novel Characteristic-Based Volume Penalization method (CBVP), discussed in this paper, employs hyperbolic forcing terms to impose general homogeneous and inhomogeneous Neumann and Robin boundary conditions. The method is flexible and can be applied to parabolic and hyperbolic evolutionary equations. In this paper it is demonstrated for viscous and inviscid flows of arbitrary Mach number. As with BPM, this method maintains rigorous control of the error through *a priori* chosen parameters for all boundary conditions.

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2 Characteristic-Based Volume Penalization

The Characteristic-Based Volume Penalization method imposes Dirichlet, Neumann, and Robin type boundary conditions by introducing localized forcing terms into the constitutive equations. For a domain containing obstacles O_m, a masking function, $\chi(\mathbf{x},t)$, is defined where

$$\chi(\mathbf{x},t) = \begin{cases} 1 & \text{if } x \in O_m, \\ 0 & \text{otherwise,} \end{cases}$$

separates the domain into a physical region and a penalized region.

Dirichlet conditions are imposed in the same fashion as with the Brinkman penalization method [7, 9]. For the boundary condition $u = u_0(\mathbf{x},t)$ on an obstacle surface $\partial O_m(\mathbf{x},t)$, the constitutive equation is modified into the penalized equation

$$\frac{\partial u}{\partial t} = (1-\chi) \times \text{RHS} - \frac{\chi}{\eta_b}(u - u_0(\mathbf{x},t)) + \chi \nu_n \frac{\partial^2 u}{\partial x_i, \partial x_i}, \tag{1}$$

with summation implied over repeated indices and where RHS is simply the physical right hand side fluxes. Convergence of the penalization parameter, as $\eta_b \to 0$, controls the error on the solution by decreasing the timescale of the forcing term [1].

The Robin boundary condition, of which the Neumann condition is a special case, has the form $a(\mathbf{x},t)u + b\partial u/\partial \mathbf{n} = g(\mathbf{x},t)$ for inward-oriented surface normal $\mathbf{n} = n_k$. It is penalized through forcing applied to the appropriate derivatives of u. The result is a hyperbolic equation,

$$\frac{\partial u}{\partial t} = (1-\chi) \times \text{RHS} - \frac{\chi}{\eta_c}\left(a(\mathbf{x},t)u + bn_k \frac{\partial u}{\partial x_k} - g\right). \tag{2}$$

With the normal defined everywhere, (2) has inward-pointing characteristics that extend perpendicular to the surface into O_m. This propagates the solution at the surface inward with a spatial growth or decay, based on g and au, that enforces the desired BC. It can easily be seen that within O_m, Robin/Neumann penalized equation (2) converge to the desired boundary condition on the timescale η_c. With $\eta_c << 1$ on the normalized problem timescale, the disparity asymptotically controls the penalization error.

3 Compressible Viscous Flows

3.1 Penalized Navier-Stokes Equations

For viscous flows, the fluid is governed by the fully compressible Navier-Stokes equations. The nondimensionalized continuity, momentum and energy equations are

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_j}{\partial x_j}, \tag{3}$$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial (\rho u_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re_a}\frac{\partial \tau_{ij}}{\partial x_j}, \tag{4}$$

$$\frac{\partial \rho e}{\partial t} = -\frac{\partial}{\partial x_j}\left[(\rho e + p)u_j\right] + \frac{1}{Re_a}\frac{\partial (u_i \tau_{ij})}{\partial x_j} + \frac{1}{(\gamma-1)}\frac{1}{Re_a Pr}\frac{\partial}{\partial x_j}\left(\mu \frac{\partial T}{\partial x_j}\right). \tag{5}$$

The acoustic Reynolds number is Re_a and Pr is the Prandtl number, and the characteristic velocity is a reference speed of sound c_0.

For the viscous benchmark problems considered in this paper, no-slip and adiabatic/heat flux conditions are imposed on velocity and temperature through (1) and (2). In order to apply these penalized boundary conditions to the constitutive equations (3-5), the equations of state are used to determine consistent penalization of the integrated variables ρ, ρu, and ρe, from the native variables u, T, and an appropriate penalized equation for ρ.

For ρ to be a passive, evolutionary condition, a CBVP Neumann condition is applied within O_m, where the target is

$$\Phi = n_k \frac{\partial \rho}{\partial x_k}\bigg|_{\partial O_m}. \tag{6}$$

This closes the penalized equations for the desired conditions on u and T without over constraining the problem. The forcing terms for the compressible Navier-Stokes equations then become

$$\frac{\partial \rho}{\partial t} = (\chi - 1)\times \mathrm{RHS} - \frac{\chi}{\eta_c}\left(n_k \frac{\partial \rho}{\partial x_k} - \Phi\right) \tag{7}$$

$$\begin{aligned}\frac{\partial \rho u_i}{\partial t} = (\chi - 1)\times \mathrm{RHS} - \chi\bigg[\frac{1}{\eta_b}\rho\,(u_i - u_{0i})\\ + \rho \nu_n \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\eta_c} u_i \left(n_k \frac{\partial \rho}{\partial x_k} - \Phi\right)\bigg]\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial \rho e}{\partial t} = (\chi - 1)\times \mathrm{RHS} - \chi\bigg[\frac{1}{\eta_c}\left(n_k \frac{\partial \rho e}{\partial x_k}\right) + \frac{\rho(u_j - u_{0j})u_j}{\eta_b}\\ - \frac{\rho u_j}{\eta_c} n_k \frac{\partial u_j}{\partial x_k} - \rho u_j \nu_n \frac{\partial^2 u_j}{\partial x_i \partial x_i} - \frac{1}{\eta_c} e\Phi - \frac{1}{\eta_c} c_v \rho q\bigg],\end{aligned} \tag{9}$$

where RHS denotes the corresponding right hand sides of equations (3 -5).

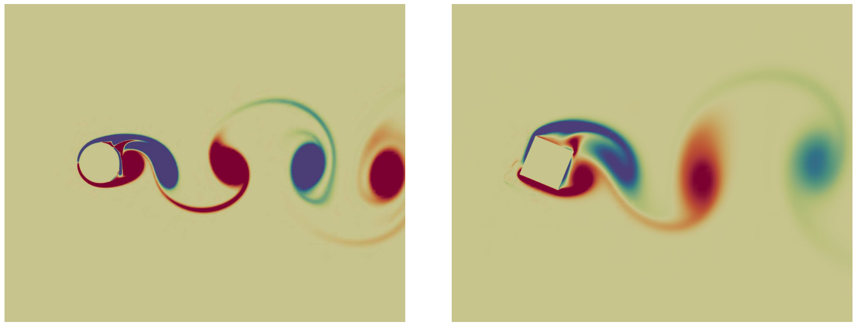

Fig. 1 The vorticity fields for flow past a circular ($Re = 1000$) and square ($Re = 150$) 2-D cylinders, demonstrating the flexibility of CBVP for arbitrary geometry.

3.2 Benchmark: 2D Cylinder Flow

To verify the efficacy of CBVP for unsteady solutions, CBVP is applied for low Reynolds number vortex shedding around a two-dimensional cylinder. For $Ma = 0.2$ and $Re = 1000$, the flow past a cylinder remains laminar but experiences vortex shedding from the trailing edge. The domain discretization and penalization parameters remain as for the pseudo-incompressible case, namely $\Omega = [-5,10] \times [-5,5]$, $\eta_b = 5 \times 10^{-3}$ and $\eta_c = 10^{-2}$. Two temperature conditions are considered: an adiabatic cylinder and constant heat flux at $\partial T/\partial \mathbf{n} = 1.5$.

Periodic vortex shedding can be seen in the laminar wake behind the cylinder in Figure 1. For laminar flows in the region of $Re \approx 1000$, the frequency is insensitive to the Reynolds number [3] and temperature driven viscosity fluctuations. The heating is therefore best seen only through the direct effect on the temperature of the fluid. Examination of the temperature profile along an arbitrary surface normal verifies that the desired heat-flux of $q = 1.5$ is properly enforced on the penalized boundary.

Time varient lift and drag coefficients C_L and C_D agree well with previous numerical results [3], though a slightly shorter shedding period can be seen. This higher frequency is reflected in a Strouhal number of $St = 0.245$, compared to $St = 0.238$ from published results [3].

To demonstrate the applicability of the method to arbitrary geometry, flow past a square cylinder is shown in Figure 1 for $Re = 150$. The masking function $\chi(x)$ and normal $\mathbf{n}(x)$ were assembled from piecewise smooth facets.

4 Compressible Inviscid Flows

4.1 Penalized Euler Equations

Inviscid flow is governed by the Euler equations, where the viscous terms are removed from (4) and (5). In this case, only the normal component of velocity will be

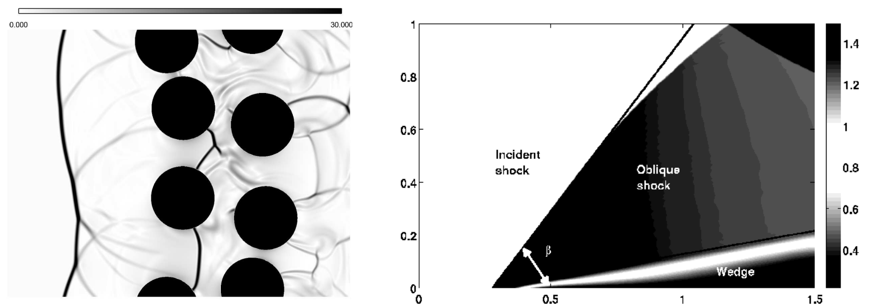

Fig. 2 Numerical Schlieren image of supersonic flow around randomly spaced multiple 2D cylinders (left) and density field of supersonic flow around the wedge with subcritical angle (right). The exact steady-state solution for the attached oblique shock wave at the wedge is drawn as the solid black line.

penalized for a no-penetration condition and the curvature of the surface must be accounted for in the boundary conditions. For consistency, the energy and momentum equations are modified based upon the penalized native variables and the equations of state. The following terms are added to the Navier-Stokes equations (7 - 9) in the inviscid limit:

$$\frac{\partial \rho}{\partial t} = \cdots + \frac{\kappa}{\eta_c} \frac{\rho u_j^\tau \rho u_j^\tau}{p}, \tag{10}$$

$$\frac{\partial \rho u_j^\tau}{\partial t} = \cdots + \frac{\kappa}{\eta_c} \frac{\rho u_j^\tau \rho u_j^\tau u_i^\tau}{p}, \tag{11}$$

$$\frac{\partial \rho e}{\partial t} = \cdots + \frac{\kappa}{\eta_c} \frac{\rho u_j^\tau u_j^\tau}{\gamma - 1} + \frac{\kappa}{\eta_c} \frac{\rho u_j^\tau \rho u_j^\tau}{p} \frac{u_i^\tau u_i^\tau - u_i^n u_i^n}{2}, \tag{12}$$

where $u^n = (u \cdot n)n$, $u^\tau = u - u^n$, $\tau = u^\tau / \|u^\tau\|$, and $\kappa = \nabla_{n\tau} \cdot n$ is a curvature along the streamline based on τ. The operator $\nabla_{n\tau}$ is a projection of gradient operator to the n, τ plane, that is $\nabla_{n\tau} = n\,(n \cdot \nabla) + \tau\,(\tau \cdot \nabla)$.

4.2 Benchmark: 2D Shocks Around Obstacles

In order to evaluate efficacy of the method, several test problems were examined: supersonic flow around multiple cylinders, and supersonic flow around the wedges with sub- and supercritical apex angles. All results showed good qualitative correspondence with published experimental and numerical results [2, 8]. For the case with supersonic flow around a wedge at a subcritical angle, there is an oblique shock inclined with some angle β. The exact, steady-state oblique shock solution is well known [6]. As shown in Figure 2, the numerical solution for a volume penalized wedge approaches the exact at steady-state. For the case with supercritical angle, a detached bow-shape shock was observed, in accordance with established results [6].

5 Conclusions

A new volume penalization method has been developed and demonstrated to extend Brinkman penalization to generalized Neumann and Robin conditions. This is accomplished through hyperbolic penalization terms whose characteristics point inward along the surface-normal direction. The process of prescribing general boundary conditons is flexible and systematic, allowing for straightforward construction of penalization schemes for arbitrary Mach and Reynolds number flows.

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