

# WAVELET-BASED COMPUTATIONAL MODELING OF WALL-BOUNDED TURBULENT FLOWS WITH LAGRANGIAN VARIABLE THRESHOLDING

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**Abstract.** The wavelet-based computational modeling of wall-bounded turbulent flows with Lagrangian variable thresholding is introduced. The eddy capturing approach exploits a wavelet-collocation parallel solver, where the flow geometry is enforced through Brinkman volume-penalization. Adaptive large-eddy simulation supplied with the one-equation localized dynamic kinetic-energy-based model is performed, where the effective turbulence resolution is locally controlled by the wavelet filtering threshold evolution. Numerical experiments are conducted for incompressible turbulent flow around a square cylinder at moderately high Reynolds number.

## 1 INTRODUCTION

The ability to identify and efficiently represent coherent flow structures, along with self-adaptiveness and de-noising, have made wavelet-based methods very useful for developing multi-resolution variable fidelity approaches to the computational modeling of turbulent flows [1]. In the wavelet-based adaptive large eddy simulation (LES) approach, the separation between resolved energetic structures and unresolved residual flow is obtained through nonlinear wavelet threshold filtering (WTF). The filtering procedure is

accomplished by applying the wavelet-transform to the unfiltered velocity field, discarding the wavelet coefficients below a given threshold  $\epsilon$  and transforming back to the physical space. This way, the turbulent velocity field is decomposed into two different parts: a coherent more energetic velocity field, which is computed, and a residual less energetic coherent/incoherent one, whose effect on the resolved motions is approximated through subgrid-scale (SGS) modeling [2].

The key role in the method is played by the thresholding level  $\epsilon$ , which explicitly defines the relative energy level of the resolved turbulent eddies and, consequently, controls the importance of the residual field to be modeled. Since its inception, the wavelet-based adaptive LES approach has been implemented with *a priori* prescribed thresholding parameter. However, when dealing with complex turbulent flows, the energy content of the dominant coherent structures to be resolved can significantly vary from region to region. In fact, the desired level of turbulence resolution can not be maintained without modifying the thresholding level. A time-dependent wavelet thresholding method has been recently introduced and successfully tested for the computational modeling of homogeneous turbulent flows, where the uniform WTF level is evolved in time according to a simple feedback control equation [3].

In this work, a new spatially and temporally varying thresholding strategy for wall-bounded flows is proposed. The procedure consists in tracking the WTF threshold within a Lagrangian frame, by directly solving the corresponding evolution equation and exploiting a path-line diffusive averaging approach. The method, which has been proposed in [4] and therein tested for forced isotropic turbulence, is applied to the computational modeling of wall-bounded flows, where the flow geometry is enforced through Brinkman volume-penalization. The wavelet-collocation/volume-penalization combined method is used in the simulation of vortex shedding flow behind an isolated stationary prism with square cross-section, immersed in a uniform fluid stream.

## 2 WAVELET-FILTERED PENALIZED GOVERNING EQUATIONS

According to the Brinkman volume-penalization approach [5], instead of solving the wavelet-filtered continuity and momentum equations in the fluid domain, with the associated no-slip boundary condition on the body surface, the following penalized equations for the filtered perturbation velocity  $\bar{u}_i^{>\epsilon}$  are solved in the entire computational domain:

$$\frac{\partial \bar{u}_i^{>\epsilon}}{\partial x_i} = 0, \quad (1)$$

$$\frac{\partial \bar{u}_i^{>\epsilon}}{\partial t} + (\bar{u}_j^{>\epsilon} + U_j) \frac{\partial \bar{u}_i^{>\epsilon}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}^{>\epsilon}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i^{>\epsilon}}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\chi_s}{\eta} (\bar{u}_i^{>\epsilon} + U_i),$$

where  $U_j$  stands for the known free-stream velocity and  $\tau_{ij} = \overline{u_i u_j}^{>\epsilon} - \bar{u}_i^{>\epsilon} \bar{u}_j^{>\epsilon}$  represents the unknown residual stresses to be modeled. The additional term at the right-hand-side of momentum equations mimics the presence of a porous body, where  $\chi_s$  stands for

the mask function associated with the obstacle domain. The parameter  $\eta$ , which has the dimension of time and reflects the fictitious porosity of the obstacle, stands for the key-parameter in the volume-penalization approach. For vanishing  $\eta$ , the solution of the penalized incompressible Navier-Stokes equations converges to the solution of the original non-penalized equations, with the global penalty error scaling as  $\eta^{1/2}$  in the fluid domain.

The filtered momentum equations are closed by means of the localized dynamic kinetic-energy-based eddy-viscosity model proposed in [2]. Namely, the deviatoric part of the residual stress tensor is approximated as  $\tau_{ij}^* \cong -2\nu_t \overline{S_{ij}^{>\epsilon}} = -2C_\nu \Delta k_{\text{sgs}}^{1/2} \overline{S_{ij}^{>\epsilon}}$ , where the turbulent eddy-viscosity is expressed in terms of the local WTF width  $\Delta$  and the SGS kinetic-energy  $k_{\text{sgs}} = \frac{1}{2}\tau_{ii}$ . The latter variable is computed by solving the following evolution equation,

$$\frac{\partial k_{\text{sgs}}}{\partial t} + (\overline{u_j^{>\epsilon}} + U_j) \frac{\partial k_{\text{sgs}}}{\partial x_j} = \nu \frac{\partial^2 k_{\text{sgs}}}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j} \left( \nu_t \frac{\partial k_{\text{sgs}}}{\partial x_j} \right) + \Pi - \epsilon_{\text{sgs}} - \frac{\chi_s}{\eta} (2k_{\text{sgs}}), \quad (2)$$

where  $\Pi = -\tau_{ij}^* \overline{S_{ij}^{>\epsilon}} \cong 2C_\nu \Delta k_{\text{sgs}}^{1/2} \overline{S_{ij}^{>\epsilon}} \overline{S_{ij}^{>\epsilon}}$  represents the SGS dissipation, which is the rate at which energy is transferred from resolved motions to unresolved ones, and  $\epsilon_{\text{sgs}}$  is the SGS energy dissipation rate approximated by  $\epsilon_{\text{sgs}} \cong C_\epsilon \frac{k_{\text{sgs}}^{3/2}}{\Delta}$ . Both the model parameters  $C_\nu$  and  $C_\epsilon$  are determined via a Germano-like dynamic procedure. It is worth noting that the eddy-viscosity coefficient  $C_\nu$  can take both signs, thus allowing for local energy backscatter ( $\Pi < 0$ ).

The ratio  $\mathcal{R}(\mathbf{x}, t) = \Pi/\mathcal{D}$ , where  $\mathcal{D} = 2\nu \overline{S_{ij}^{>\epsilon}} \overline{S_{ij}^{>\epsilon}}$  stands for the resolved viscous dissipation, can be used as a measure of the turbulence resolution. A prescribed value for this quantity (say  $\mathcal{R}_{\text{goal}}$ ) determines the desired level at which the more energetic structures are resolved and the effect of SGS residual motions is modeled.

### 3 LAGRANGIAN VARIABLE THRESHOLDING

As demonstrated by previous wavelet-based LES studies, the SGS dissipation increases with the WTF level. Therefore, the two-way energy transfer between resolved and unresolved motions can be controlled by varying this parameter. In fact, a decrease of the threshold  $\epsilon$  results in the local grid refinement with the subsequent rise of the resolved viscous dissipation, while an increase of it leads to mesh coarsening, which results in the growth of the local SGS dissipation.

Therefore, for wall-bounded flows, the wavelet-based adaptive LES method can be drastically improved by exploiting a spatially-varying threshold based on the explicit control of the turbulence resolution. The basic idea is to locally vary  $\epsilon$  wherever the fraction of modeled dissipation deviates from an a-priori defined magnitude. In order to vary the WTF level in a physically consistent fashion, by following the turbulent flow structures as they evolve in space and time, the Lagrangian representation of this parameter is considered. Based upon the Lagrangian path-line diffusive averaging approach [6], the following

evolution equation for the variable threshold  $\epsilon(\mathbf{x}, t)$ ,

$$\frac{\partial \epsilon}{\partial t} + (\bar{u}_j^{>\epsilon} + U_j) \frac{\partial \epsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \nu_\epsilon \frac{\partial \epsilon}{\partial x_j} \right) + \mathcal{F}_\epsilon, \quad (3)$$

is solved along with the filtered momentum equation (1). The additional diffusion term at the right-hand-side of (3) is added to guarantee the smoothness of the solution [4].

The above equation is solved starting from an uniform WTF level  $\epsilon(\mathbf{x}, 0) = \epsilon_0$ , with suitable conditions on the boundary of the computational domain. In a practical calculation, the variable threshold is constrained to take value in a prescribed interval,  $\epsilon_{\min} \leq \epsilon \leq \epsilon_{\max}$ . The upper limit is chosen to maintain a certain numerical accuracy, which would unavoidably deteriorate for too high WTF levels [7]. On the other side, for too low thresholds, the coherent vortex simulation (CVS) regime would be achieved, where the modeled SGS dissipation would be negligible.

The forcing term  $\mathcal{F}_\epsilon$  in the evolution equation (3) is proportional to the difference between the actual and prescribed values of the dissipation ratio. In this study, the following feedback forcing is used

$$\mathcal{F}_\epsilon(\mathbf{x}, t) = - \left( \mathcal{R}^{1/2} - \mathcal{R}_{\text{goal}}^{1/2} \right) \frac{\epsilon}{\tau_\epsilon},$$

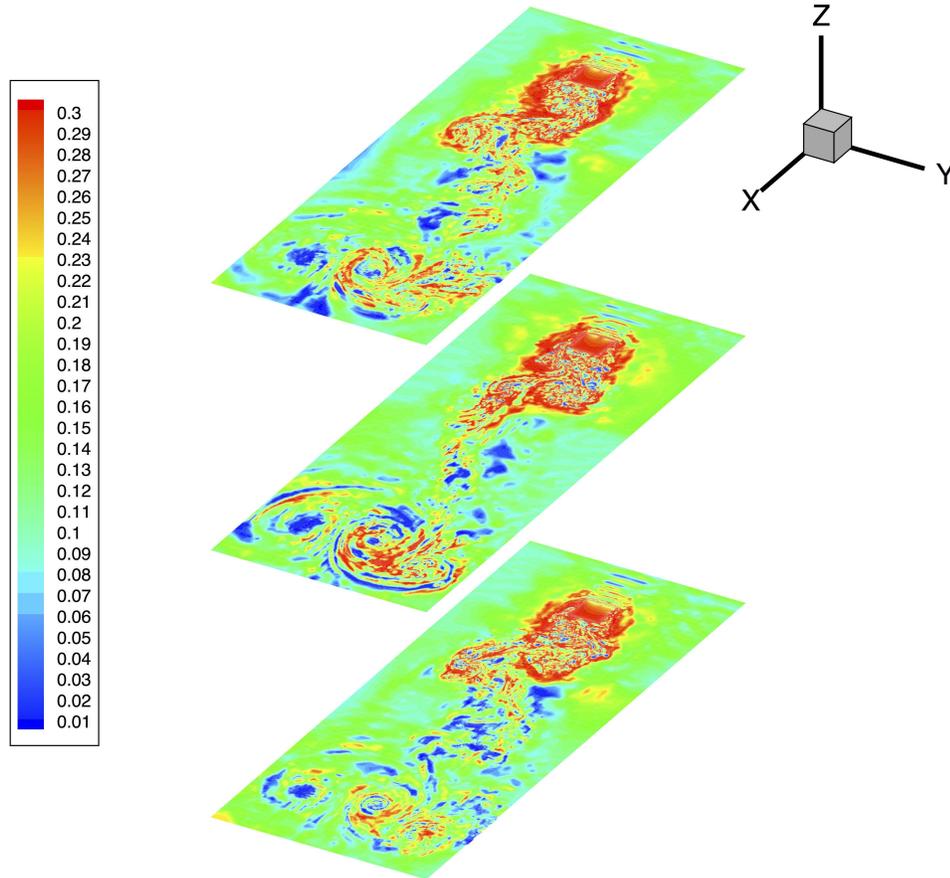
where  $\tau_\epsilon$  represents a relaxation time parameter. Similarly to the previous time-varying threshold studies [3], the local time-scale is set to the characteristic rate-of-strain, specifically,  $\tau_\epsilon(\mathbf{x}, t) = (2\nu/\mathcal{D})^{1/2}$ , is used. Furthermore, the forcing term is turned off in regions of negative SGS dissipation, where energy backscatter occurs. This way, it holds

$$\mathcal{F}_\epsilon(\mathbf{x}, t) = \begin{cases} - \left( \frac{1}{2\nu} \right)^{1/2} \left( \Pi^{1/2} - \mathcal{R}_{\text{goal}}^{1/2} \mathcal{D}^{1/2} \right) \epsilon, & \text{if } \Pi > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and, as expected, the threshold  $\epsilon$  tends to increase where  $0 < \Pi/\mathcal{D} < \mathcal{R}_{\text{goal}}$ , and to decrease where  $\Pi/\mathcal{D} > \mathcal{R}_{\text{goal}}$ .

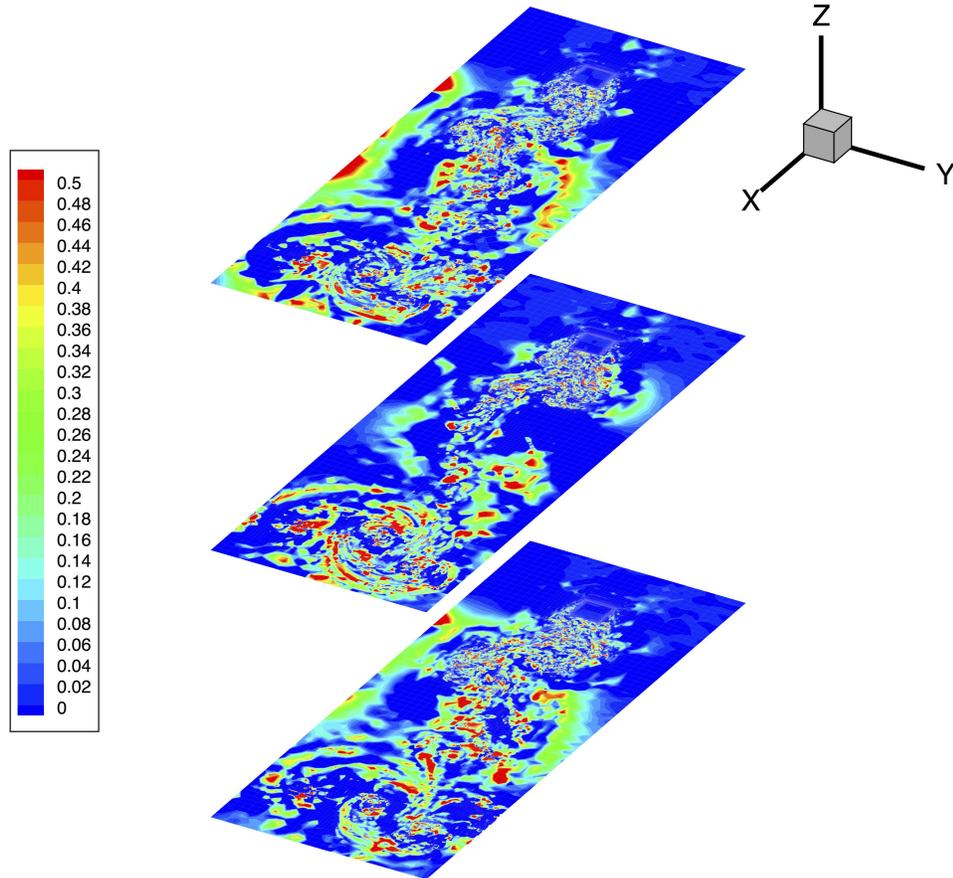
## 4 RESULTS

The hybrid wavelet-collocation/volume-penalization method with Lagrangian variable thresholding is applied to the simulation of incompressible turbulent flow past a square cylinder at moderate Reynolds number, which is  $\text{Re} = UL/\nu = 2000$ , where  $U$  is the free-stream velocity and  $L$  is the side of the square section. The initial velocity and SGS energy fields correspond to the adaptive LES solution of the same flow conducted with constant and uniform threshold that is  $\epsilon_0 = 0.10$ . The variable threshold is allowed to vary in the range  $0.01 < \epsilon < 0.3$ , starting from  $\epsilon(\mathbf{x}, 0) = 0.10$ , with imposed Neumann boundary conditions at the body surface. The goal value for the dissipation ratio is prescribed as  $\mathcal{R}_{\text{goal}} = 0.25$ . The simulation is conducted until a statistically converged solution is obtained.



**Figure 1:** Contour map of the instantaneous WTF level ( $0.01 < \epsilon < 0.3$ ) at three different planes along the spanwise direction ( $z = -1.5, 0$  and  $1.5$ ).

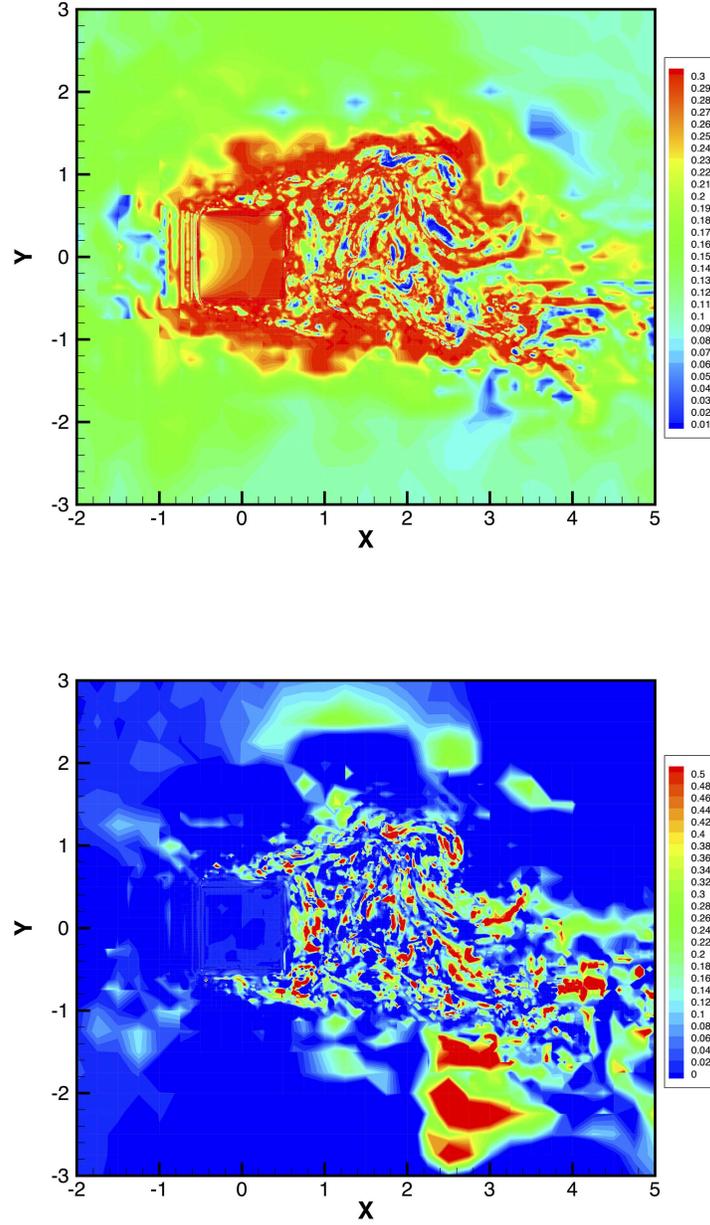
The instantaneous WTF level field  $\epsilon$  is illustrated in Figure 1, where the contour maps at three different planes along the spanwise direction ( $z = -1.5, 0$  and  $1.5$ ) are reported. Note that the two-dimensional obstacle occupies the space region determined by  $-0.5 < x < 0.5$ ,  $-0.5 < y < 0.5$ . The corresponding modeled to dissipation ratio  $\mathcal{R}$  is depicted in Figure 2, where the “blue zones” correspond to flow regions in which backscatter occurs ( $\mathcal{R} < 0$ ), and “red zones” those ones in which  $\mathcal{R} > 0.5$ . By making a comparison between these two pictures, it appears that the feedback mechanism works as expected. For instance, in flow regions where the SGS dissipation is too high (with respect to the goal value), the WTF level is maintained at its minimum possible value. On the contrary, in regions where the SGS dissipation is too low, the threshold has got its maximum value as happens, in particular, close to the body surface, where the numerical



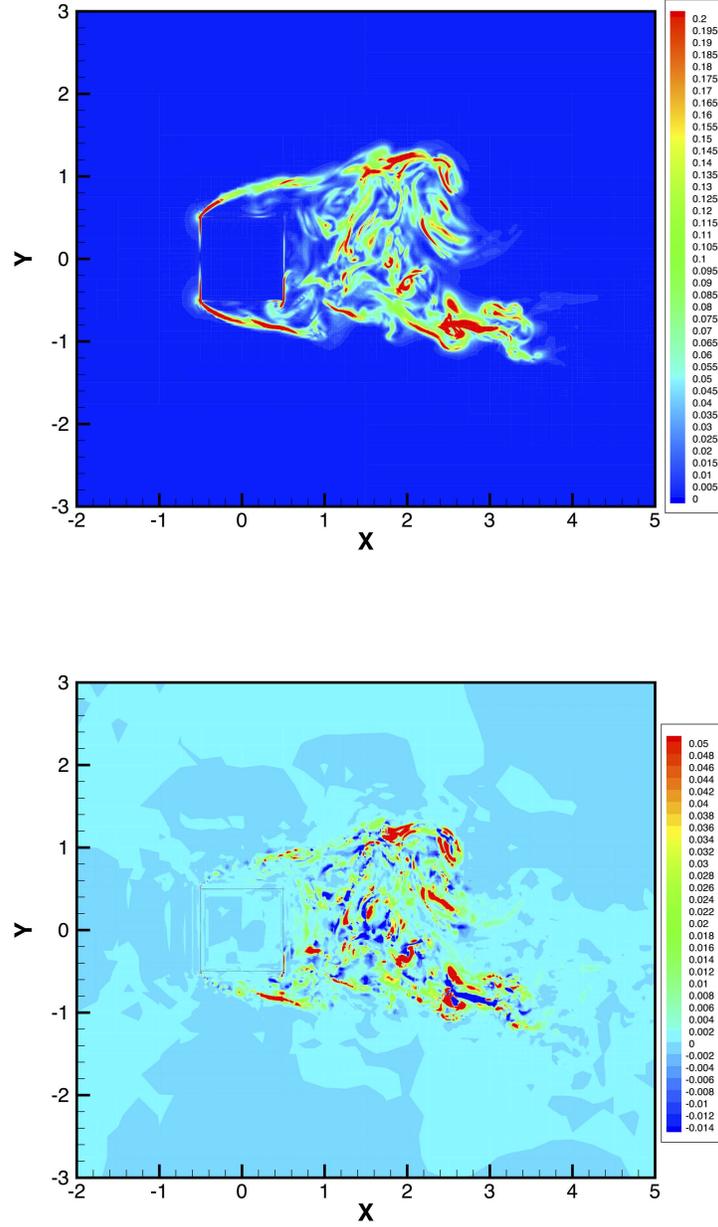
**Figure 2:** Contour map of the instantaneous modeled to resolved dissipation ratio ( $0 < \mathcal{R} < 0.5$ ) at three different planes along the spanwise direction ( $z = -1.5, 0$  and  $1.5$ ).

grid is finer. This is apparent from the close-up views in the neighborhood of the obstacle at the midplane ( $z = 0$ ), which are reported in the following figures. The contour maps of the threshold  $\epsilon$  and the ratio  $\mathcal{R}$  are shown on the top and the bottom of Figure 3, respectively. The flow structure corresponding to the separated shear layer on the sides of the cylinder is evident by looking at Figure 4, where the resolved viscous dissipation  $\mathcal{D}$  (top) and the modeled SGS dissipation  $\Pi$  (bottom) are drawn.

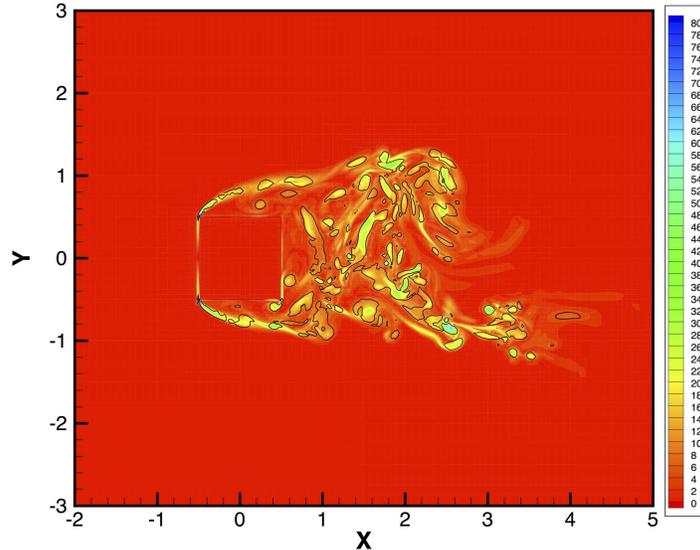
For the sake of validation, the solution is further examined by plotting in Figure 5 the contour map of the vorticity vector norm  $\omega$ . In this picture, the main vortical structures are identified according to the *Q-criterion* by drawing the isolines of  $Q = 6$ , where  $Q$  stands for the second invariant of the velocity-gradient tensor. These results are in excellent agreement with the results from the non-adaptive numerical solution presented in [8].



**Figure 3:** Contour map of the instantaneous WTF level ( $0.01 < \epsilon < 0.3$ ) (top) and modeled to resolved dissipation ratio ( $0 < \mathcal{R} < 0.5$ ) (bottom) at the midplane ( $z = 0$ ), in the neighborhood of the obstacle ( $-0.5 < x < 0.5$ ,  $-0.5 < y < 0.5$ ).



**Figure 4:** Contour map of the instantaneous resolved viscous dissipation ( $0 < \mathcal{D} < 0.2$ ) (top) and modeled SGS dissipation ( $-0.014 < \Pi < 0.05$ ) (bottom) at the midplane ( $z = 0$ ), in the neighborhood of the obstacle ( $-0.5 < x < 0.5$ ,  $-0.5 < y < 0.5$ ).



**Figure 5:** Contour map of the instantaneous vorticity norm ( $0 < \omega < 80$ ), along with the isolines of  $Q = 6$ , at the midplane ( $z = 0$ ), in the neighborhood of the obstacle ( $-0.5 < x < 0.5$ ,  $-0.5 < y < 0.5$ ).

## 5 CONCLUDING REMARKS

The present method of physics-based varying thresholding fully exploits the spatial and temporal intermittency of turbulence, while overcoming the major limitation for wavelet-based multi-resolution techniques that make use of a constant and uniform thresholding criterion. The Lagrangian variable thresholding represents a new strategy for the wavelet-based adaptive LES of wall-bounded turbulent flows. A two-way feedback mechanism between the modeled dissipation and the numerical resolution is provided, where a prescribed level of turbulence resolution for the ongoing simulation is maintained. This study demonstrates that the present adaptive hybrid methodology for modeling bluff-body flows is feasible, accurate and efficient.

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