Mathematical Modelling of Hydraulic Fracturing and Related Problems

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Stages of the hydraulic fracturing modeling

- Fracture propagation
  - interaction of fracturing fluid with the elastic reservoir
  - fluid leak-off and interaction with pore fluid
  - rock toughness, confining stresses, etc.

- Fluid flow
  - fluid-proppant mutual influence
  - proppant transport, settlement
  - bridging of proppant

- Fractured well production
  - production forecast
  - multiple wells interaction

- Hydraulic fracture control
  - Determination of the size and location of the fracture
Fracture growth in a poroelastic medium
Khristianovich-Geertsma-de-Klerk (KGD)

Flow in the fracture:

Mass balance: \( \frac{\partial w}{\partial t} + \frac{\partial Q}{\partial x} = -q_L \)

Carter’s formula: \( q_L = \frac{2C_L}{\sqrt{t - t_{\text{exp}}(x)}} \)

Poiseuille’s formula: \( Q = -\frac{w^3}{12\mu} \frac{\partial p}{\partial x} \)

Elasticity:

\[
 w(t, x) = \frac{4}{\pi E'} \int_0^l (p(t, \xi) - \sigma) B(x, \xi) d\xi, \quad E' = \frac{E}{1 - \nu^2}, \tag{1}
\]

where \( E \) and \( \nu \) are the Young’s modulus and Poisson’s ratio,

\[
 B(x, \xi) = \ln \left| \frac{\sqrt{l^2 - x^2} + \sqrt{l^2 - \xi^2}}{\sqrt{l^2 - x^2} - \sqrt{l^2 - \xi^2}} \right|. \tag{2}
\]

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**Perkins-Kern-Nordgren (PKN)**

- **Flow in the fracture:**
  - **Mass balance:** \( \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = -H q_L \),
  - **Carter’s formula:** \( q_L = \frac{2C_L}{\sqrt{t - t_{\text{exp}}(x)}} \),
  - **Poiseuille’s formula:** \( Q = -\frac{\pi E'}{512\mu} \frac{\partial w_{\text{max}}}{\partial x} \)

- **Elasticity:**
  \[
  w(t, z) = \frac{4(p(t) - \sigma)}{E'} \sqrt{\frac{H^2}{4} - z^2}, \quad A = \frac{1}{4}\pi H w_{\text{max}}
  \]
State of the Art

Mostly used hydraulic fracturing models:

- KGD, PKN, penny-shaped models (Classic)
- P3D, Planar3D
- 3D models

Some drawbacks:

- Influence of pore pressure to the strain is not properly accounted
- Leak-off requires additional assumptions (a subject of discussions)
- Infinite fluid pressure at the fracture tip

Review of modern results:

Fracture in a poroelastic medium\textsuperscript{2}

**Biot’s equations of poroelasticity**

\[
\begin{align*}
\text{div } \tau &= 0, \quad \tau = \lambda \text{tr } \mathbf{E}(\mathbf{u}) \mathbf{I} + 2\mu \mathbf{E}(\mathbf{u}) - \alpha p \mathbf{I} \\
S_e \frac{\partial p}{\partial t} &= \text{div} \left( \frac{k_r}{\eta_r} \nabla p - \alpha \frac{\partial \mathbf{u}}{\partial t} \right)
\end{align*}
\]

\[\Gamma_R : p = p_\infty, \quad \tau \langle \mathbf{n} \rangle = -\sigma_\infty \mathbf{n}\]

\[\Gamma_s : u_y = 0, \quad v = 0, \quad p_y = 0\]

\[\Gamma_c : \tau \langle \mathbf{n} \rangle = -U \mathbf{n}\]

\[\Gamma_c : \frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left( \frac{v^3}{3\eta_r} \frac{\partial U}{\partial x} \right) - q_l; \quad \left. \frac{\partial U}{\partial x} \right|_{y=0, x=0^\mp} = \frac{q(t)}{2} = \frac{Q(t)}{2H}\]

Non-Stationary Self-Similar Solution\(^3\)

The representation of solution:

\[
\begin{align*}
  u(t, x, y) &= t^{1/6} U(\xi, \eta), & \xi(t, x, y) &= \frac{x}{\sqrt{t}} \\
  v(t, x, y) &= t^{1/6} V(\xi, \eta), & \eta(t, x, y) &= \frac{y}{\sqrt{t}} \\
  p(t, x, y) &= t^{-1/3} P(\xi, \eta), & L_i(t) &= \gamma_i \sqrt{t}, & i = r, l
\end{align*}
\]

Assumptions:

- **Injected flow rate:** \( q = q_0 t^{-1/3} \)
- **Velocity** of fracture tips: \( v_{tip} = v_0 t^{-1/2} \).

Solution Method: Fracture Growth Criterion

- cohesive forces
- $p_f$
- $2w$
- $\sigma_{coh}$
- cohesive zone

- $\sigma_{coh}$
- $\sigma_c$
- $G_c/2$
- $w_m$
- $w_c$
- $w$
Case studies: Common parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus, $E$</td>
<td>17 GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio, $\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>Reservoir Permeability, $k_r$</td>
<td>100 mD</td>
</tr>
<tr>
<td>Biot Coefficient, $\alpha$</td>
<td>0.75</td>
</tr>
<tr>
<td>Storativity, $S_e$</td>
<td>$1.5 \times 10^{-8}$ Pa$^{-1}$</td>
</tr>
<tr>
<td>Closure Stress, $\sigma_\infty$</td>
<td>3.7 MPa</td>
</tr>
<tr>
<td>Reservoir Pressure, $p_\infty$</td>
<td>0 MPa</td>
</tr>
<tr>
<td>Reservoir Fluid Viscosity, $\eta_r$</td>
<td>$10^{-3}$ Pa · sec</td>
</tr>
<tr>
<td>Fracture Fluid Viscosity, $\eta_f$</td>
<td>$10^{-3}$ Pa · sec</td>
</tr>
<tr>
<td>Rate per Unit Height, $q$</td>
<td>$10^{-2}$ m$^2$/sec</td>
</tr>
</tbody>
</table>
Case Studies: Pressure and fracture aperture

Pressure and half-width over the boundary $y = 0$
Case Study: Non-Uniform Closure Stress

\[ \sigma_{\infty}^l = 10.1 - 10.8 \text{ MPa} \]

\[ \sigma_{\infty}^r = 10 \text{ MPa} \]
Case Study: Non-Uniform Closure Stress

Pressure and fracture aperture over the boundary $y = 0$. 
Case Studies: Non-Uniform Permeability

\[
k_r = 10^{-14} \text{ m}^2
\]

\[
k_r = 10^{-16} \text{ m}^2
\]
Case Studies: Non-Uniform Permeability

Pressure and fracture aperture over the boundary $y = 0$. 
Case Studies: Non-Uniform Permeability

Pressure and fracture aperture over the boundary $y = 0$. 
Case Studies: Non-Uniform Permeability

Pressure and fracture aperture over the boundary $y = 0$. 
Case Studies: Non-Uniform Permeability

Pressure and fracture aperture over the boundary $y = 0$. 

![Graph showing pressure and fracture aperture over the boundary $y = 0$.](image-url)
Case Studies: Non-Uniform Permeability

Pressure and fracture aperture over the boundary \( y = 0 \).
Inhomogeneity of fracturing fluid
Statement of the problem

- The fracture design supposes a stage of pumping of increasing concentration of proppant
- Model of the fracture growth should take into account
  - Non-uniform distribution of proppant in a crack
  - Instabilities due to different viscosities of fluid components
  - Mutual influence of proppant distribution and fracture opening
Pseudo two-speed model

Mass conservation laws:

\[
\begin{aligned}
\frac{\partial (\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{u}) &= -Q_{lf}, \\
\frac{\partial (wd)}{\partial t} + \nabla \cdot (wd \vec{u}) &= 0
\end{aligned}
\]

\[
u = cu_p + (1 - c)u_f, \quad s = \bar{d} \equiv \frac{1}{cH} \int_{\Gamma_p} \dd y.
\]
Pseudo two-speed model

Averaged system:

\[
\begin{align*}
\frac{\partial (cw)}{\partial t} + \frac{\partial (cw u_p)}{\partial x} &= -v_l c \left( 1 - \frac{s}{s_l} \right) + Q, \\
\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} &= -v_l \left( 1 - \frac{cs}{s_l} \right), \\
\frac{\partial s}{\partial t} + u_p \frac{\partial s}{\partial x} &= \frac{s}{w} \left( v_l \left( 1 - \frac{s}{s_l} \right) - \frac{Q}{c} \right).
\end{align*}
\]
Micro-polar and visco-plastic fluids

Examples of micro-polar and visco-plastic fluids: fracturing fluid with proppant, drilling fluid, blood, etc.

In the model we take into account fluid-particle and particle-particle interaction: $\mathbf{v}$ is the suspension velocity, $\mathbf{\omega}$ is the angular speed of micro-rotations, $p$ is the pressure, $B = \nabla \mathbf{v} + \varepsilon \cdot \mathbf{\omega}$ is the strain tensor. The stress tensor is

$$T = -pI + 2\mu_1 B_s + 2\mu_2 B_a + \tau_\ast \frac{B}{|B|}, \quad \text{if} \quad B \neq 0$$

Right: dimensionless flow rate vs. dimensionless pressure gradient. The curves from top to bottom correspond to increase of viscosity $\mu_2$. The top curve $\tau_\ast = \mu_2 = 0$ is the Navier-Stokes.

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Inflow to a horizontal multiply-fractured well
Inflow to a horizontal multiply-fractured well

Physical parameters:
- \( p \) — pressure
- \( \rho = \text{const} \) — density
- \( m = m(p) \) — porosity
- \( \varepsilon \) — elastic capacity
- \( k = k(x, y, z) \) — rock permeability
- \( \mu \) — fluid viscosity

Proposed 2D-model accounts for

- Arbitrary net of fractures with different conductivities of segments
- Variable in space and time physical parameters of the reservoir
- Arbitrary boundary conditions over the outer boundary an at the borehole (given pressure or flow rate)
Optimization of fractures geometry

Well production ($10^3 \text{ m}^3$) versus time (days): (a) — four short rare fractures; (b) — two long fractures; (c) — four dense short fractures

Pressure distribution at $t = 60$ days.
An “arbitrary” fracture net

An “arbitrary” set of fractures and the computational mesh

Well production ($10^3$ m$^3$) versus time (days).
An “arbitrary” fracture net

Pressure distribution: $k = 1\text{ mD}$, $p|_O = 100\text{ atm}$, given pressure over outer boundary
Hydraulic fracture control
Phase shift between waves of velocity and pressure

By the direct numerical modelling it is shown that under non-stationary injection/suction of fluid into the fractured well, the phase shift between the velocity $v$ and pressure $p$ is observed.

This observation allows determination of $L$ by the solution of the parametric optimization problem.
After the pumping stop, the fracture is closing. 1 is an invasion zone, 2 virgin zone, \( \varphi \) is the streaming potential, \( \sigma \) is the conductivity of filtrating electrolytes:

\[
Q = -\lambda_{11} \nabla p - \lambda_{12} \nabla \varphi, \quad J = -\lambda_{21} \nabla p - \lambda_{22} \nabla \varphi,
\]

\[
\lambda_{11} = k/\eta, \quad \lambda_{22} = \sigma, \quad \lambda_{12} = F \sqrt{\lambda_{11} \lambda_{22}}
\]

Electrokinetic coefficients \( \lambda_{ij} \) are discontinuous on the invasion front. Electric field \( E \) is perpendicular to the hydraulic fracture and grows as the rations \( \lambda_{11}^{(2)}/\lambda_{11}^{(1)} \) and \( \lambda_{22}^{(1)}/\lambda_{22}^{(2)} \) decrease. This corresponds to decrease of viscosity and electric conductivity of invading fluid.

Related problems: cerebral haemodynamics
Cerebral arteriovenous malformations

An arteriovenous malformation is a tangle of arteries and veins that affects normal cerebral blood circulation.

In the brain, damage occurs through 4 mechanisms:

- Theft of blood to neighbouring nerve tissue.
- Brain haemorrhage.
- Compression of surrounding nervous tissue.
- Making the flow of cerebrospinal fluid (CSF) more difficult.

\[6\text{Pictures source: http://www.neuros.net}\]
Cerebral arteriovenous malformations

Embolisation of arteriovenous malformation.

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6 Pictures source: http://www.neuros.net
Experimental data

In vivo measurements of pressure and velocity in brain vessels were done in Meshalkin Research Institute Of Blood Circulation Pathology.  

Volcano ComboMap

VP-diagram in the afferent of AVM

A – before the embolisation,
B – after the embolisation.

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Experimental data

Typical graphs of *in vivo* measured velocity and pressure of the blood flow in a feeding artery (afferent) and a draining vein of an AVM.  
*Left*: Time series.  
*Right*: $vp$-diagram in the afferent of the AVM before and after the embolisation, and in the draining vein.
# Numerical experiments

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length scale</td>
<td>$L = 10$ cm</td>
</tr>
<tr>
<td>Porosity</td>
<td>$m_0 = 0.2$</td>
</tr>
<tr>
<td>Permeability</td>
<td>$k = 2 \cdot 10^{-9}$ cm$^2$</td>
</tr>
<tr>
<td>Time</td>
<td>$T = 1$ s</td>
</tr>
<tr>
<td>Vessel radius</td>
<td>$R = 0.1$ cm</td>
</tr>
<tr>
<td>Fluid viscosity</td>
<td>$\mu = 2.4 \cdot 10^{-2}$ g/(cm s)</td>
</tr>
<tr>
<td>Elastic capacity</td>
<td>$\varepsilon = 3.75 \cdot 10^{-7}$ cm$^2$/g</td>
</tr>
<tr>
<td>Walls permeability</td>
<td>$\kappa = 10^{-4}$ cm$^2$ s/g</td>
</tr>
</tbody>
</table>

**Input velocity** $v(O)$

**Output pressure** $p(E)$

![Diagram of arterial and venous trees of vessels](image)
Numerical results

Computational results for velocity and pressure. 

*Left:* Time series at the roots of the arterial ($O$) and the venous ($E$) trees, and in the afferent.

*Right:* $v_p$-diagram in the afferent and at the root $E$ of the venous tree before and after the embolisation.
Further development

- Modelling of 3D fractures in inhomogeneous reservoir;
- Interpretation of the mini frac tests;
- Estimation of the fracture conductivity after the pressure drop;
- Dynamics of multi fracturing;
- Filtration of multiphase fluids and interaction with HF;
- Dynamics of non-Newtonian fluids in tubes and fractures;
THANK YOU FOR ATTENTION!