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September 17, 2021.

Re: Report on the Doctor of Philosophy in Mathematics thesis “Twisted representations of quantum toroidal algebras and their applications” by Roman Gonin.

The thesis is devoted to the study of the representation theory of the quantum toroidal algebra of type gl(1) (which I will denote by **E**) via bosonization techniques. The algebra **E** depends on two parameters and appears in many applications, e.g. as the Hall algebra on the elliptic curve (Burban-Schiffman, Schiffman-Vasserot), algebra of correspondences on the equivariant K-theory of Hilbert scheme of points in the complex plain (Feigin – Tsymbaliuk, Schiffman-Vasserot), as the spherical double affine Hecke algebra (Feigin-E. Feigin-Jimbo-Miwa-M); it plays a crucial role in the construction of a refined topological vertex (Awata-Feigin-Shiraishi) and of an integrable system (Feigin-Kojima- Shiraishi-Watanabe, Feigin-Jimbo-Miwa-M). The algebra **E** is the simplest quantum toroidal algebra and is a very important object of study in modern mathematics.

The algebra **E** has a set of generators labeled by the integer lattice in a real plane and a two-dimensional center. The group SL(2,Z) acts on the lattice and the center; it extends to the action on **E** by automorphisms. It is known (Schiffman) that **E** is generated by generators labeled by integer points with coordinates (a,1) , (a,0), and (a,-1) where a is an arbitrary integer. These generators are called parallel generators; they satisfy standard commutation relations.

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However, there are no good formulas to express all other generators through parallel ones. Because of the large group of automorphisms, one can define many other sets of generators. For example, the generators corresponding to the points (1,a), (0,a), (-1,a) are called perpendicular generators; they correspond to the 90 degree rotation matrix in SL(2,Z).

The algebra **E** has a distinguished remarkable module **F** which is call the Fock module. *In parallel generators*, the module **F** is identified with the space of symmetric polynomials, where **E** acts by Macdonald operators and by multiplication operators. In particular, it is an **E**-module of highest weight, and the action is given by explicit formulas in terms of partitions. The parallel Cartan generators corresponding to the lattice points (a,0) commute. *In perpendicular generators*, the same module **F** is identified with the free boson, where generators act by coefficients of explicit vertex operators. The perpendicular Cartan generators corresponding to lattice points (0,a) form a Heisenberg algebra. The parallel realization is convenient for various combinatorial studies. The perpendicular realization is suitable for the study of the integrable systems.

The principal result of the thesis is the construction of the same Fock module **F** in the generators corresponding to the automorphisms coming from the *arbitrary* elements of SL(2,Z). Since the automorphisms are very non-trivial, one has to use some clever tricks to find a bypass to the answer. The actual proof goes through the semi-infinite wedge construction, the lifting of the action to double affine Hecke algebra, and then taking a rather difficult inductive limit.

The final formulas are quite interesting and confirm a conjecture of Gorsky-Negut. Namely, the action of **E** is given in terms of quantum affine algebra gl(n), where n depends on the choice of an element of SL(2,Z). The space **F** is identified with the vacuum module of the quantum affine gl(n). The generators of quantum affine gl(n) act in the vacuum module by well-known vertex operators, and these operators are used to write the action of generators of **E**.

In my opinion, this is a fundamental result which improves our understanding of quantum toroidal gl(1) and its representations. I hope that now we have a chance to study tensor products of Fock modules with different twists which should have many implications for integrable systems. It also would be interesting to extend the result of this thesis to quantum toroidal gl(n) with arbitrary n. Thus this thesis may serve as a starting point for many future works.

In addition to the main result, the thesis contains a few other curious formulas. One such formula is a new bosonization of the deformed W-algebras of type gl(n). The author also

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provides realizations of the so called "twisted deformed W-algebras". While the immediate applications of that are not clear to me at the moment, I believe one day these formulas will be useful.

To summarize, the results of this thesis are new, non-trivial, and interesting. I have no doubts that this thesis satisfies all the requirements for the degree of the Doctor of Philosophy in Mathematics.

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