Skolkovo Institute of Science and Technology

Jury Member Report – Doctor of Philosophy thesis.

Name of Candidate: Vadim Prokofev

PhD Program: Mathematics and Mechanics

Title of Thesis: Integrable hierarchies of nonlinear differential equations and many-body systems

Supervisor: Professor Anton Zabrodin

Name of the Reviewer:

I confirm the absence of any conflict of interest (Alternatively, Reviewer can formulate a possible conflict)	**Date: 03-03-2022**

The purpose of this report is to obtain an independent review from the members of PhD defense Jury before the thesis defense. The members of PhD defense Jury are asked to submit signed copy of the report at least 30 days prior the thesis defense. The Reviewers are asked to bring a copy of the completed report to the thesis defense and to discuss the contents of each report with each other before the thesis defense.

If the reviewers have any queries about the thesis which they wish to raise in advance, please contact the Chair of the Jury.

Reviewer's Report

Reviewers report should contain the following items:

- Brief evaluation of the thesis quality and overall structure of the dissertation.
- The relevance of the topic of dissertation work to its actual content
- The relevance of the methods used in the dissertation
- The scientific significance of the results obtained and their compliance with the international level and current state of the art
- The relevance of the obtained results to applications (if applicable)
- The quality of publications

The summary of issues to be addressed before/during the thesis defense

Report on the Ph.D. thesis by V. Prokofev at Skoltech and HSE, Moscow

V. Prokofev's thesis is dedicated to the study of the correspondence between the dynamics of poles of solutions of integrable hierarchies (such as KP, matrix KP, 2d Toda lattice) and Calogero-Moser and Ruijsenaars-Schneider systems as well as their spin versions. This continues previous works of a number of authors, notably I. Krichever. The thesis is based on five papers by Prokofev joint with his adviser A. Zabrodin.

Let me summarize the contents of these works.

In arXiv:1907.06621, Prokofev and Zabrodin consider solutions of the 2D Toda lattice hierarchy which are trigonometric functions of the "zeroth" time $t_0 = x$. It is known that their poles move as particles of the trigonometric Ruijsenaars-Schneider model. They extend this correspondence to the level of hierarchies: the dynamics of poles with respect to the m-th hierarchical time t_m, $m \in \mathbb{Z}$, of the 2D Toda lattice hierarchy is shown to be governed by the Hamiltonian which is proportional to the m-th Hamiltonian $\mathrm{tr}L^m$ of the Ruijsenaars-Schneider model, where L is the Lax matrix.

In arXiv:1910.00434, Prokofev and Zabrodin consider solutions of the matrix KP hierarchy that are trigonometric functions of the first hierarchical time $t_1 = x$ and establish the correspondence with the spin generalization of the trigonometric Calogero-Moser system on the level of hierarchies. Namely, the evolution of poles x_i and matrix residues at the poles $a_i^\alpha b_i^\beta$ of the solutions with respect to the k-th hierarchical time of the matrix KP hierarchy is shown to be given by the Hamiltonian flow with the Hamiltonian which is a linear combination of the first k higher Hamiltonians of the spin trigonometric Calogero-Moser system with coordinates x_i and with spin degrees of freedom $a_i^\alpha b_i^\beta$. By considering evolution of poles according to the discrete time matrix KP hierarchy they also introduce the integrable discrete time version of the trigonometric spin Calogero-Moser system.

In arXiv:2102.03784 Prokofev and Zabrodin consider solutions of the KP hierarchy which are elliptic functions of $x = t_1$. It is known that their poles as functions of t_2 move as particles of the elliptic Calogero-Moser model. They extend this correspondence to the level of hierarchies and find the Hamiltonian H_k of the elliptic Calogero-Moser model which governs the dynamics of poles with respect to the k-th hierarchical time. The Hamiltonians H_k are obtained as coefficients of the expansion of the spectral curve near the marked point in which the Baker-Akhiezer function has an essential singularity.

In arXiv:2103.00214 Prokofev and Zabrodin consider solutions of the 2D Toda lattice hierarchy which are elliptic functions of the zeroth time $t_0 = x$. It is known that their poles as functions of t_1 move as particles of the elliptic Ruijsenaars-Schneider model. The goal of this paper is to extend this correspondence to the level of hierarchies. The authors show that the Hamiltonians which govern the dynamics of poles with respect to the m-th hierarchical times t_m and $\bar{t}_m$ of the 2D Toda lattice hierarchy are obtained from expansion of the spectral curve for the Lax matrix of the Ruijsenaars-Schneider model at the marked points.

Finally, in arXiv:2103.07357 Prokofev and Zabrodin consider solutions of the matrix KP hierarchy that are elliptic functions of the first hierarchical time $t_1 = x$. It is known that poles x_i and matrix residues at the poles $\rho_i^{\alpha\beta} = a_i^\alpha b_i^\beta$ of such solutions as functions of the time t_2 move as particles of spin generalization of the elliptic Calogero-Moser model (elliptic Gibbons-Hermsen model). In this paper the authors establish the correspondence with the spin elliptic Calogero-Moser model for the whole matrix KP hierarchy. Namely, they show that the dynamics of poles and matrix residues of the solutions with respect to the k-th hierarchical time of the matrix KP hierarchy is Hamiltonian with the Hamiltonian H_k obtained via an expansion of the spectral curve near the marked points. The Hamiltonians are identified with the Hamiltonians of the elliptic spin Calogero-Moser system with coordinates x_i and spin degrees of freedom a_i^α, b_i^β.

I think this is a good Ph.D. thesis that is a welcome addition to the integrable systems literature, which complements and completes the pioneering work of I. Krichever and others 40 years ago and more recently.

Here are some minor comments on the text.

1. Formula (1.24) contains a misprint.

2. Is the coefficient 2 for the potential needed in (1.24)? Of course, for classical systems it does not matter since the coupling constant can be renormalized, but I believe the standard normalization is without 2. Same comment for (1.17).

3. Chapter 3, line 3,4: the label 4.3 repeats several times in the same sentence, probably it is a misprint.

4. Formula (3.10). I am not sure what is meant by a “formal Lie algebra”. The set defined by (3.10) is not a Lie algebra, some vanishing conditions are needed to make it so (I am sure the author understands this perfectly well, just uses an unusual terminology which may be confusing).

Sincerely,
Pavel Etingof
Professor of Mathematics
MIT

Provisional Recommendation

☒ *I recommend that the candidate should defend the thesis by means of a formal thesis defense*

☐ *I recommend that the candidate should defend the thesis by means of a formal thesis defense only after appropriate changes would be introduced in candidate's thesis according to the recommendations of the present report*
☐ *The thesis is not acceptable and I recommend that the candidate be exempt from the formal thesis defense*