Skolkovo Institute of Science and Technology

SACHDEV-YE-KITAEV MODEL IN THE PRESENCE OF THE QUADRATIC PERTURBATION.

Doctoral Thesis
by

Aleksey Lunkin

DOCTORAL PROGRAM IN PHYSICS

Supervisor
Assistant Professor, Konstantin Tikhonov

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I hereby declare that the work presented in this thesis was carried out by myself at Skolkovo Institute of Science and Technology, Moscow, except where due acknowledgement is made, and has not been submitted for any other degree.

Candidate (Aleksey Lunkin)

Supervisor (Assistant Prof. Konstantin Tikhonov)

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Abstract

In this work, we study the Sachdev-Ye-Kitaev (SYK) model in the presence of quadratic perturbation. The original SYK model describes a system of strongly interacted electrons. This model is analytically solvable i.e., we can find the behaviour of the Green function for all ranges of times. The Green function demonstrates non-Fermi liquid behaviour. The perturbation "tries" to restore Fermi-liquid properties, i.e., suppress the interaction's influence. In this thesis, we describe the scenario of this suppression. Firstly, we have shown that due to strong fluctuations, the property of the SYK model is stable with respect to small perturbation. Increasing the strength of the perturbation, we can observe the suppression of the fluctuation due to phase transition. This transition is connected with the appearance of the bound states of the Hamiltonian of the 1D quantum mechanical problem, which effectively decreases the fluctuation's behaviour. We also studied behaviour of the green function in the case of the big number of bound states.

Publications

Main author

1. A.V. Lunkin, K.S. Tikhonov, and M.V. Feigel'man. Sachdev-ye-kitaev model with quadratic perturbations: The route to a non-fermi liquid. *Phys. Rev. Lett.*, 121(23):236601, 2018

2. A. V. Lunkin, A. Yu. Kitaev, and M. V Feigel'man. Perturbed sachdev-ye-kitaev model: a polaron in the hyperbolic plane. *Phys. Rev. Lett.*, 125(19): 196602, 2020

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Glossary

FL Fermi liquid. 9

JT Jackiw–Teitelboim. 10

NFL non-Fermi liquid. 24

OTOC out-of-time ordered correlation function. 10

RG renormalization group. 26

SYK Sachdev-Ye-Kitaev. 9

Chapter 1

Introduction

The Sachdev-Ye-Kitaev (SYK) model describes a system of strongly interacted fermions. Let us briefly discuss the reasons to study such systems.

We want to start with the history of fermion systems. The most famous fermion particle is an electron. Metals contain many electrons, so their properties, such as conductivities (electric and thermal) and heat capacity at sufficiently low temperatures, are determined by the behavior of electrons. The theory which describes quantitative properties of metal was suggested by Enrico Fermi [14]. He considered many non-interacting fermions in the free space and showed that the thermodynamic properties of such systems are similar to those of metals.

This result sounds wonderful. Since the school, we know about Coulomb interaction between charged particles. Why is this interaction not important? Lev Landau obtained the answer to this question. He showed that weak interaction does not change Fermi's results qualitatively. The model proposed by Fermi is called Fermi gas. The theory suggested by Landau is called Fermi Liquid (FL) [23]. According to this theory, we can describe low-energy excitation of the interacted systems using a non-interacting system of "quasi-particles"—the difference between parameters of particles and quasi-particles caused by the interaction.

The Landau Fermi liquid theory is not the end of the study of fermionic systems. As was mentioned, interaction should be weak and repulsive. Weak attractive interactions lead to superconductivity [7]. In this case, we can also describe the system's behavior using quasi-particles, but their spectrum (dependence of energy on the momentum) will be different.

What should we expect in the case of a strong interaction? There is no such classification of phases of matter in this regime, and the analytical description of such theories is very poor. Several years ago, Alexei Kitaev suggested SYK model [1]. The model has several interesting properties: behavior of the fermionic green function is not Fermi liquid-like, the Lyapunov exponent is maximal for the interacted systems, the low-energy effective action of the model is similar to the action of the so-called JT gravity [18, 31, 22]. Since the properties of these models are interesting, it is reasonable to ask a question about their stability with respect to perturbations.

The original SYK model describes a zero-dimensional system with $N \gg 1$ degenerate levels occupied by Majorana fermions with random interaction. The characteristic scale of the interaction is J. We assume that perturbation lifts this degeneracy and tries to restore FL. The characteristics scale of the perturbation is Γ. In our analysis, we will show that for $\Gamma \ll \frac{J}{N}$ SYK behavior is stable, whereas in the opposite case, FL behavior is restored.

1.1 Literature review

As was mentioned above, the original model was presented by Kitaev [1]. The description of the mean-filed solution, fluctuation around this solution, and calculation of the out-of-time ordered correlation function (OTOC) could be found in [21, 26]. This result shows that the fluctuation is important for sufficiently large times or low temperatures, and we should not limit by the saddle-point approximation. The approach which lets to obtain results for such times was initially developed by Bagrets, Altland and Kamenev [4]. They have shown that the model at zero temperature could be described using Liouville quantum mechanics. This result was extended to the case of non-zero temperature in the work [28] using methods of conformal field theory. Work [22] also extends the results to finite temperatures and provides quasi-classical intuition: the low-energy action of the SYK model is equivalent to the action of the particle with an imaginary charge which diffuses over the hyperbolic plane.

The extension of the SYK model was studied in a large amount of work. The most important work for us is [30] where SYK model was extended to the higher dimensions. The author considered the lattice of quantum dots, the dynamic inside dots are covered by the SYK Hamiltonian, and there is tunnelling between them. The authors showed that in the limit of small tunnelling conductivity and high temperature, the conductivity of the model is proportional to the inverse temperature, and one can observe the crossover to the Fermi-liquid regime with decreasing of the temperature. All these results were obtained in the saddle-point approximation.

It was mentioned above that saddle-point approximation is not applicable for sufficiently large times or law temperatures. So the question about crossover is not answered in the strong fluctuation regime. We will try to answer this question for the simplified model in this thesis.

1.2 Thesis objectives

This thesis aims to study the influence of the quadratic (in fermions) perturbation on the property of the SYK model for various ranges of temperatures. We will show that the presence of quadratic perturbation leads to the phase transition. These results are important in the study of fermionic systems with strong interaction.

1.3 Research methodology

The main method of research used in this work is a functional integral approach. We use the technique developed in the work [4] to study the case of the zero temperature. The essence of one is to map the problem to the 1D quantum mechanical problem known as Liouville quantum mechanics. We also extend this technique to the case of non-zero temperature using results of the work [22].

1.4 Scientific novelty

All results obtained in the work are new. They were published in the refereed journal [25, 24]. Particularly, the question about the influence of the fluctuation on the crossover to the Fermi-liquid behavior was firstly asked at our work. The stability of the model at a small but non-zero area of parameters was shown firstly in our work [25]. The existence of the phase transition between the region with strong fluctuations and fermi-liquid behavior was first observed in the work [24].

1.5 Thesis structure

In the section "SYK model" one can find introduction in the topic and review of the technique from the [4]. In the section "SYK model with quadratic perturbations: the route to a non-Fermi-liquid." we will consider the effect of the perturbation for the system with zero temperature using perturbation theory. In the section "Perturbed Sachdev-Ye-Kitaev model: a polaron in the hyperbolic plane." we are considering the self-consistent approach to the study of the problem with non-zero temperature. The last section is conclusion. We also included two appendices with technical details.

Chapter 2

SYK model

In this chapter we will consider the pure SYK model and its main properties. The material from this chapter is taken from [21, 26, 4]

2.1 The SYK Hamiltonian and action

The Hamiltonian of the SYK model has the following form:

$$H = \frac{1}{4!} \sum_{ijkl=0}^{N} J_{ijkl} \chi_i \chi_j \chi_k \chi_l. \tag{2.1}$$

Here χ_i are Majorana fermion operator. They satisfies the following commutation relation:

$$\{\chi_i, \chi_j\} = 1. \tag{2.2}$$

There is an extension of the SYK model for the case of real fermions [16] but it is not sufficient for our conclusions.

Couplings J_{ijkl} are Gaussian random variables with zero mean and the following variance:

$$\overline{J_{ijkl}^2} = \frac{6J^2}{N^3}. \tag{2.3}$$

Here N is a number of different operators. These coupling are also anti-symmetric in indices.

Our main interest is a fermionic Green function. To calculate one, let us introduce an action for the problem using imaginary times. The action has a form:

$$S_{SYK} = \int_0^\beta d\tau \left(\frac{1}{2} \sum_i \chi_i(\tau) \partial_\tau \chi_i(\tau) + H \right). \tag{2.4}$$

We will perform several transform to make the action simpler. The first step is to perform average over disorder. We will take an average of the partition function instead taking average of free-energy as we plan to work above temperature of the glass transition [21]. More over, the existence of this transition is a debatable question [17]. After average, the action will be:

$$S_{SYK} = \int_0^\beta d\tau_1 d\tau_0 \left(\frac{1}{2} \sum_i \chi_i(\tau_1) \sigma(\tau_1, \tau_0) \chi_i(\tau_0) - \frac{J^2}{8N^3} \left[\sum_i \chi_i(\tau_1) \chi_i(\tau_0) \right]^4 \right). \tag{2.5}$$

Here we introduced $\sigma(\tau_1, \tau_0) \equiv \delta'(\tau_1 - \tau_0)$. We also would like to work with field $G(\tau_1, \tau_0)$ defined as:

$$G(\tau_1, \tau_0) = -\frac{1}{N} \sum_l \chi_l(\tau_1) \chi_l(\tau_0). \tag{2.6}$$

To make this field independent we need to introduce a Lagrangian multiplier $\Sigma(\tau_1, \tau_0)$ using these new fields we can write an action of the problem as:

$$\begin{aligned} S_{SYK} = \frac{N}{2} \int_0^\beta & d\tau_1 d\tau_0 \Big(\{ -\sigma(\tau_1, \tau_0) + \Sigma(\tau_1, \tau_0) \} G(\tau_1, \tau_0) \\ & - \frac{J^2}{4} G^4(\tau_1, \tau_0) + \Sigma(\tau_1, \tau_0) \sum_l \chi_l(\tau_1) \chi_l(\tau_0) \Big). \end{aligned} \tag{2.7}$$

As one can see, the action is quadratic with respect to fermion operators and we can take functional integral over fields χ. As a result the action for the problem would be written in terms of fields G and Σ and has the following form:

$$S_{SYK} = \frac{N}{2} tr\ln(-\hat{\Sigma}) + \frac{N}{2}\int_0^\beta d\tau_1 d\tau_0 \left(\{-\sigma(\tau_1,\tau_0) + \Sigma(\tau_1,\tau_0)\} G(\tau_1,\tau_0) - \frac{J^2}{4} G^4(\tau_1,\tau_0) \right). \tag{2.8}$$

As $S \sim N$ and $N \gg 1$ we can use mean-field approximation to take the functional integral.

2.2 Mean-field equations and symmetries.

For this chapter we would like to change our time variable τ to a dimensional one. So we would introduce new coordinate θ defined as

$$\theta \equiv \frac{2\pi\tau}{\beta}. \tag{2.9}$$

It is also convenient to rescale fields:

$$G(\tau_1,\tau_0) \equiv \left(\tfrac{2\pi}{\beta J}\right)^{1/2} G(\theta_1,\theta_0), \qquad \Sigma(\tau_1,\tau_0) \equiv J^2 \left(\tfrac{2\pi}{\beta J}\right)^{3/2} \Sigma(\theta_1,\theta_0),$$
$$\sigma(\tau_1,\tau_0) \equiv J^2 \left(\tfrac{2\pi}{\beta J}\right)^{3/2} \sigma(\theta_1,\theta_0). \tag{2.10}$$

As a result we can write an action (2.8) in the form:

$$S_{SYK} = \frac{N}{2} tr\ln(-\hat{\Sigma}) + \frac{N}{2}\int_0^{2\pi} d\theta_1 d\theta_0 \left(\{-\sigma(\theta_1,\theta_0) + \Sigma(\theta_1,\theta_0)\} G(\tau_1,\tau_0) - \frac{1}{q} G^q(\theta_1,\theta_0) \right). \tag{2.11}$$

Here we also introduced a parameter q in our case $q = 4$ but sometimes it is reasonable to consider other value of this parameter [26, 21]. In the limit $N \gg 1$ we can consider a mean-field approximation as fluctuations are suppressed as $\frac{1}{N}$. These equations have a form:

$$\int d\theta \Sigma(\theta_1,\theta) G(\theta,\theta_0) = -\delta(\theta_1-\theta_0),$$
$$\Sigma(\theta_1,\theta_0) = G^{q-1}(\theta_1,\theta_0) + \sigma(\theta_1,\theta_0). \tag{2.12}$$

In the limit $\beta J \gg 1$ we can neglect by $\sigma(\theta_1, \theta_0)$ for $\theta_1 - \theta_0 \gg \frac{1}{\beta J}$. As a result, the mean-field equations in the low-temperature limit will be:

$$\Sigma(\theta_1, \theta_0) = G^{q-1}(\theta_1, \theta_0) \qquad \int d\theta, \Sigma(\theta_1, \theta) G(\theta, \theta_0) = -\delta(\theta_1 - \theta_0). \tag{2.13}$$

This equation has quite big symmetry group. Particularly, we can consider any diffeomorphism of the unit circle i.e periodic, monotonic function $\varphi(\theta)$ ($\varphi(\theta) = \varphi(\theta+2\pi)$). Using this function and arbitrary solution of the above equation $G(\theta_1, \theta_0)$ we can construct a new solution using the following transform:

$$G(\theta_1, \theta_2) \mapsto [\varphi'(\theta_1)\varphi'(\theta_0)]^{\Delta} G(\varphi(\theta_1), \varphi(\theta_2)). \tag{2.14}$$

Here $\Delta = \frac{1}{q}$.

Among all solution we need translation invariant which has the form:

$$G_s(\theta_1, \theta_0) = -b^{\Delta} |\theta_{10}^2|^{-\Delta} sgn(\theta_{10}) \quad \text{where} \quad \theta_{10} = 2 \sin\left(\frac{\theta_1 - \theta_0}{2}\right), \quad b = \frac{(1 - 2\Delta)\tan(\pi\Delta)}{2\pi}. \tag{2.15}$$

The symmetry group of this solution is much smaller than the symmetry group of the equation. This group can be described by the transformations which can be written as:

$$e^{i\varphi(\theta)} \equiv \frac{a e^{i\theta} + b}{\bar{b} e^{i\theta} + \bar{a}}, \qquad |a|^2 - |b|^2 = 1. \tag{2.16}$$

As we see the symmetry group of equations is smaller then the symmetry group of equations (2.13). It is an indicator of the presence of the Goldstone mode in the problem. Our case is slightly different. We used approximated equations so our symmetry is not exact. It reminds us the case of σ-model (for introduction in the topic see [12]). So, in this problem we should not take an integral over all function G. We should consider only ones which is solution of the saddle-point equations (2.13), such fields can be obtained by the application of diffeomorphism to the saddle-point solution (2.14). The fluctuation along this manifold is enhanced by the parameter

βJ [21]. As a result for $\beta J \sim N$ we can not suppose that fluctuation are small and we should consider full action for the fields from these manifolds. Fields parametrized by the function $\varphi(\tau)$ and we can rewrite action for the fields from this manifold using this function. The action has the form:

$$S_{soft} = -\frac{\alpha_S N}{J} \int\limits_0^\beta Sch\left\{e^{i\varphi(\tau)}, \tau\right\} d\tau, \qquad Sch\left\{f(\tau), \tau\right\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.17}$$

Here $\alpha_S \approx 0.05$ could be found numerically. This result was obtained in works [21, 26].

2.3 Green function for $\beta = \infty$.

In this section we will consider the case with $\beta = \infty$. The idea of this section was presented at the work [4]. For zero temperature we should not consider diffeomorphisms of the circle instead we should consider monotonic re-paramtrizations of the real axis: $\tau \mapsto f(\tau)$. The action is defined as:

$$S_{soft} = -\frac{\alpha_S N}{J} \int\limits_{-\infty}^{\infty} Sch\left\{f(\tau), \tau\right\} d\tau. \tag{2.18}$$

The field G is parameterised as:

$$G_{\tau_1,\tau_0}[f(\tau)] = -b^\Delta sgn(\tau_1 - \tau_0)\left[\frac{f'(\tau_1)f'(\tau_0)}{J^2|f(\tau_1) - f(\tau_0)|^2}\right]^\Delta. \tag{2.19}$$

It is convenient to introduce new field defined as:

$$f(\tau)' \equiv e^{\phi(\tau)}. \tag{2.20}$$

We are interested in the behaviour of the Green function of fermions which is the average of the field G. So we need to calculate the following average (we assume that $\tau 1 > \tau_0$):

$$\langle (G_{\tau_1,\tau_0}[f(\tau)])^n \rangle = \left(-\frac{b^\Delta}{J^{2\Delta}} \right)^n \int \mathcal{D}\phi \left[\frac{e^{\phi(\tau_1)} e^{\phi(\tau_0)}}{| \int_{\tau_0}^{\tau_1} e^{\phi(\tau)} d\tau |^2} \right]^{n\Delta} \times \\ \exp\left\{ -\frac{M}{2} \int_{-\infty}^{\infty} (\phi')^2 d\tau \right\} \tag{2.21}$$

here we introduced $M \equiv \alpha_S N/J$.

This functional integral could be calculated exactly using 1D quantum mechanics but firstly we need to fix a gauge to not worry about zero modes of the functional integral. One can note that theory is symmetric under transform $\phi \mapsto \phi + a$. We can fix a gauge assuming that $\phi(\tau_0) = 0$. Now we need to use next auxiliary integral:

$$a^{-p} = \frac{1}{\Gamma(p)} \int_0^\infty dx x^{p-1} e^{-ax}. \tag{2.22}$$

Here $\Gamma(p)$ is the gamma-function. Using the auxiliary integral we can re-write our average in the form:

$$\langle (G_{\tau_1,\tau_0}[f(\tau)])^n \rangle = \left(-\frac{b^\Delta}{J^{2\Delta}} \right)^n \int_0^\infty \frac{dx}{\Gamma(2n\Delta)} x^{2n\Delta - 1} \times \\ \int \mathcal{D}\phi e^{n\Delta\phi(\tau_1)} e^{n\Delta\phi(\tau_0)} \exp\left\{ -\int_{-\infty}^{\infty} \frac{M}{2} (\phi')^2 d\tau - x \int_{\tau_0}^{\tau_1} e^{\phi(\tau)} d\tau \right\}. \tag{2.23}$$

To take this functional integral we would like to introduce a Hamiltonian:

$$\hat{H}_x = -\frac{\partial_\phi^2}{2M} + x e^\phi. \tag{2.24}$$

The quantum mechanical problem with this Hamiltonian is called Liouville quantum mechanics. The spectrum of this Hamiltonian is continuous . The wave function could be written as:

$$\langle \phi | p, x \rangle = \sqrt{\tfrac{2p \sinh(2p\pi)}{\pi}} K_{2ip}(\sqrt{8xMe^\phi}) = \\ \sqrt{\tfrac{2p \sinh(2p\pi)}{\pi}} \int\limits_{-i\infty+0}^{i\infty+0} \tfrac{ds}{2\pi i} \Gamma\left(s - ip\right) \Gamma\left(s + ip\right) \left[2xMe^\phi \right]^{-s}. \tag{2.25}$$

It is wave function with quantum number p for the potential with parameter x. We use Mellin transform to write the last form in the above expression. The energy of such functions is $E_p = \frac{p^2}{2M}$. This function are eigenfunction of the above Hamiltonian and we also normalized them. As a result we can write average using evolution operator of this Hamiltonian in the following form:

$$\langle (G_{\tau_1,\tau_0}[f(\tau)])^n \rangle = \left(-\frac{b^\Delta}{J^{2\Delta}}\right)^n \int_0^\infty \frac{dx}{\Gamma(2q\Delta)} x^{2n\Delta-1} \langle 0,0| e^{n\Delta\hat{\phi}} \hat{U}_x(\tau_1,\tau_0)|0\rangle. \quad (2.26)$$

Here $|0\rangle = \delta(\phi)$, we use this representation due to choice of the gauge. The evolution operator has a form:

$$\hat{U}_x(\tau_1,\tau_0) = \int_0^\infty \frac{dp}{2\pi} |p,x\rangle\langle p,x| e^{-\frac{p^2}{2M}(\tau_1-\tau_0)}. \quad (2.27)$$

Taking almost all integrals we obtain an integral representation for our average:

$$\langle (G_{\tau_1,\tau_0}[f(\tau)])^n \rangle = \left(-\frac{b^\Delta}{J^{2\Delta}}\right)^n \int_0^\infty dp \frac{p\sinh(2\pi p)}{2\pi^2} \Gamma^2\left(n\Delta + ip\right) \Gamma^2\left(n\Delta - ip\right) e^{-\frac{p^2}{2M}(\tau_1-\tau_0)}. \quad (2.28)$$

We see that there is the scale M in the problem. For $\tau_1 - \tau_0 \ll M$, this average could be calculated using saddle-point approximation over p. For the opposite case, we see that the leading contribution comes from the area with $p \ll 1$, as a result:

$$\langle (G_{\tau_1,\tau_0}[f(\tau)])^n \rangle \approx \left(-\frac{b^\Delta}{J^{2\Delta}}\right)^n \Gamma^4\left(n\Delta\right) \frac{1}{\sqrt{2\pi}} \left(\frac{\tau}{M}\right)^{-3/2} \quad (2.29)$$

As a conclusion of this part we see that SYK model has two qualitatively different regimes: short time behaviour, defined by mean field equations and long-time behaviour which is defined by strong fluctuations. Both regimes demonstrates non-Fermi-liquid behaviour. In the next chapters we will consider the stability of these properties.

Chapter 3

SYK model with quadratic perturbations: the route to a non-Fermi-liquid.

The material for this part of the thesis is taken from our paper [25].

3.1 Introduction

The SYK model seems to be a very promising starting point to approach a theory of non-Fermi-liquid ground state. Few problems arise, however: i) the absence of a quadratic term in the Hamiltonian makes pure SYK Hamiltonian unrealistic for electronic systems; ii) original SYK model contains Majorana fermions, which are quite scarce in Nature (see however few relevant proposals in Refs.[9, 29, 8]); iii) most interesting properties of a non-Fermi-liquid state are those related to transport phenomena, while SYK is a random-matrix-type model without spatial coordinates. Quite a number of recent publications address the issues listed above [6, 11, 20, 30]. In particular, generalisation of the SYK model for complex fermions was developed in Refs. [11, 30]. A sequence of SYK "quantum dots" connected by weak (quadratic) tunnelling was considered in Refs. [20, 30], making it possible to define and study transport quantities like resistance, thermal resistance, etc; see also very recent extensive study in the same direction [10].

However, all (known to us) studies of stability of SYK behavior w.r.t. to quadratic perturbations, indicate its runaway instability. As it was shown in Refs. [6, 20, 30, 10] in the framework of the self-consistent approximation, the scaling dimension of the SYK_2 perturbation is negative when estimated within the conformal limit, corresponding to the time-scales $1/J \ll \tau \ll t_c \sim M$. The papers [30, 10] demonstrate an interesting non-Fermi-liquid behavior in the intermediate temperature region $T^* < T \ll J$, but still obtain Fermi-liquid behavior in the lowest T range below T^*.

In the present chapter we reconsider the problem of the SYK_4 stability w.r.t. quadratic perturbations, going beyond the saddle-point approximation. We study fermionic Green function in the region $\tau \gg t_c$ by means of perturbation theory in the amplitude of SYK_2 terms, using the infra-red asymptotic solution [26, 4] as a starting point. We show analytically that a weak SYK_2 perturbation does not change the $G(\tau) \propto 1/\tau^{3/2}$ asymptotics of the Green function, but simply renormalizes the coefficient. This result proves the existence of a *domain of stability*, with a non-zero area in the parameter space of Hamiltonians, where a non-Fermi-liquid is realized as a ground-state.

3.2 The model

We consider the model defined by the following Hamiltonian

$$H = \frac{1}{4!}\sum_{i,j,k,l} J_{i,j,k,l}\chi_i\chi_j\chi_k\chi_l + \frac{i}{2!}\sum \Gamma_{i,j}\chi_i\chi_j. \tag{3.1}$$

where χ_i are Majorana fermions and all indices run from 1 to N. The matrix elements J_{ijkl} and $\Gamma_{i,j}$ are fully anti-symmetric and independent random Gaussian variables with zero mean and the variances $\langle J_{ijkl}^2\rangle = \frac{3!J^2}{N^3}$, $\langle \Gamma_{ij}^2\rangle = \frac{\Gamma^2}{N}$. The functional integral representation of this theory is described by the action $S = -\frac{N}{2}(S_1 + S_2)$ with two contributions[26, 4]:

$$S_1 = tr\log(\partial_\tau - \Sigma_{\tau\tau'}) + \int d\tau d\tau' \left(\frac{J^2}{4}G^4_{\tau\tau'} - \Sigma_{\tau\tau'}G_{\tau'\tau}\right), \tag{3.2}$$

and

$$S_2 = \int d\tau d\tau' \frac{\Gamma^2}{2} G^2_{\tau\tau'}. \tag{3.3}$$

In the limit $N \gg 1$ the mean-field analysis is appropriate and the corresponding saddle-point equations read

$$\partial_\tau G_{\tau\tau'} - \int d\tau'' \Sigma_{\tau\tau''} G_{\tau''\tau'} = \delta(\tau - \tau'), \tag{3.4}$$

$$\Sigma_{\tau\tau'} = J^2 G^3_{\tau\tau'} + \Gamma^2 G_{\tau\tau'}. \tag{3.5}$$

We are going to study corrections to the SYK model Green function $G(\tau)$ assuming dimensionless parameter $\gamma = \Gamma/J$ to be small. Within applicability range of the saddle-point Eqs. (3.4), (3.5), the scaling dimension of the perturbation is negative, $\Delta_\gamma = -\frac{1}{2}$. As a result, $G^{(0)}_{SYK}(\tau) \propto (J\tau)^{-1/2}$ (the mean-field solution at $\gamma = 0$) is unstable w.r.t. the perturbation: at $\tau \geq \tau^* \sim 1/J\gamma^2$ it is replaced by the usual Fermi-liquid behavior $G(\tau) \propto (J\gamma\tau)^{-1}$. On the other hand, at sufficiently long times $t \gg t_c$, soft re-parametrization modes [1, 21, 26] become relevant and the Green function of the pure SYK_4 model acquires different scaling [4]

$$G(\tau) = \frac{\Gamma^4(\frac{1}{4})}{\sqrt{2MJ}\pi^{5/4}} \left(\frac{M}{\tau}\right)^{\frac{3}{2}}. \tag{3.6}$$

For sufficiently weak perturbation $\gamma \ll 1/\sqrt{N}$, the crossover timescale τ^* becomes larger than t_c and loses its relevance: the analysis of the SYK solution stability should now be performed using the asymptotic behaviour (3.6) as a starting point.

First of all we need to write action for new problem which take perturbations into account. This action has a form:

$$S_{soft} = S_\phi + S_2[\phi] \quad \text{where} \quad S_\phi = -\frac{\alpha_S N}{J} \int_0^\beta Sch\left\{f(\tau), \tau\right\} d\tau,$$

$$S_2[\phi] = \frac{N\Gamma^2}{2} b^{2\Delta} \int d\tau_1 d\tau_0 \left[\frac{f'(\tau_1) f'(\tau_0)}{J^2 |f(\tau_1) - f(\tau_0)|^2}\right]^{2\Delta}. \tag{3.7}$$

We have limited ourselves by fields that lie in the manifold of soft modes as only they determine the behavior of the system at small energies.

3.3 Perturbation theory.

First-order correction to the Green's function $G(\tau)$ due to the quadractic term S_2 in the action can be found (see Appendix A for more details) in a straightforward way as follows (notation $\langle .. \rangle_0$ means the average over ϕ field with the action S_ϕ, see Eq. (3.7)):

$$\delta G(\tau) = -\langle G_{\tau,0}[\phi] S_2[\phi] \rangle_0 + \langle G_{\tau,0} \rangle_0 \langle S_2 \rangle_0. \tag{3.8}$$

Substituting here Eq. (3.3), we find that the first term of Eq. (3.8) contains an average (over $\phi(\tau)$ fluctuations) of the product of three functionals like (2.19), with the time arguments $0, \tau$ and τ_1, τ_2, where further integration over $\tau_{1,2}$ is implied. Functional integration over $\phi(\tau)$ with the action (3.7) should be performed separately in 6 different time regions with the following order of time arguments:

1	2	3	4	5	6
$\tau_2, \tau_1, 0, \tau$	$\tau_2, 0, \tau_1, \tau$	$\tau_2, 0, \tau, \tau_1$	$0, \tau_2, \tau_1, \tau$	$0, \tau_2, \tau, \tau_1$	$0, \tau, \tau_2, \tau_1$

Domains 1 & 6 have trivial structure and their contributions are canceled completely by the second term in Eq. (3.8). Combining other contributions with the corresponding parts of the second term in Eq.(3.8), we find

$$\delta G(\tau) = \frac{N\sqrt{MJ}}{4\pi^{5/4}} \gamma^2 \left[\sum_{i=2}^{5} f_i(\frac{\tau}{2M}) - f_Z(\frac{\tau}{2M}) \right]. \tag{3.9}$$

where functions $f_i(x)$ (for $i = 2, 3, 4, 5$) and $f_Z(x)$ are defined and calculated in the Appendix A. In total, in the long-time limit $\tau \gg M$ we have:

$$\delta G(\tau) = cN\sqrt{MJ}\gamma^2 (\tau/M)^{-\frac{3}{2}}. \tag{3.10}$$

with $c \approx 108$. Comparing with Eq. (3.6), we find $\delta G/G \approx 3.7 JMN\gamma^2$ As a result

$$\delta G/G \approx 0.081 N^2 \gamma^2. \tag{3.11}$$

Equation (3.11) demonstrates that relevant parameter of the perturbation theory in the infrared limit is actually $\gamma N \equiv b$, and perturbation of the SYK_2 type only modifies the numerical prefactor in $G(\tau)$.

3.4 Conclusion

Schematically, our results for the zero-temperature phase diagram of the combined SYK_4 - SYK_2 model are shown in the Fig. 3-1. We emphasize somewhat unusual scaling limit of large N that is employed here. Namely, we consider $N \gg 1$ as some finite number, but we neglect exponentially small many-body level spacing $\epsilon_{MB} \sim 2^{-N/2}$. Then our results demonstrate the presence of a *phase transition* between fully chaotic non-Fermi-liquid (NFL) ground state realized at $b \equiv \gamma N < b_c$, and Fermi-liquid ground state existing at $b > b_c$, with $b_c \sim 1$. We emphasize that the corresponding critical value of the amplitude of the quadratic perturbation Γ equals $\Gamma_c = b_c J/N$. In other terms, the effect of this perturbation in the infrared limit is much stronger than one could naively expect considering its effect at short times $t \leq t_c$ where relevant Γ scales as $1/\sqrt{N}$.

Note that $1/t^{3/2}$ long-time asymptotics of the Green function in pure SYK model can be understood (We thank the referee of our paper for mentioning to us this relation.) as a result of the square-root edge singularity of the full many-body DoS [5, 15], together with chaotic non-structured nature of matrix elements that enter Lehman expansion for the Green function. Then, the phase transition we found upon increase of quadratic perturbation b can be understood as a transition to non-chaotic state, with matrix elements aquiring nontrivial structure leading to Fermi-liquid type of behavior $G(t) \sim 1/t$.

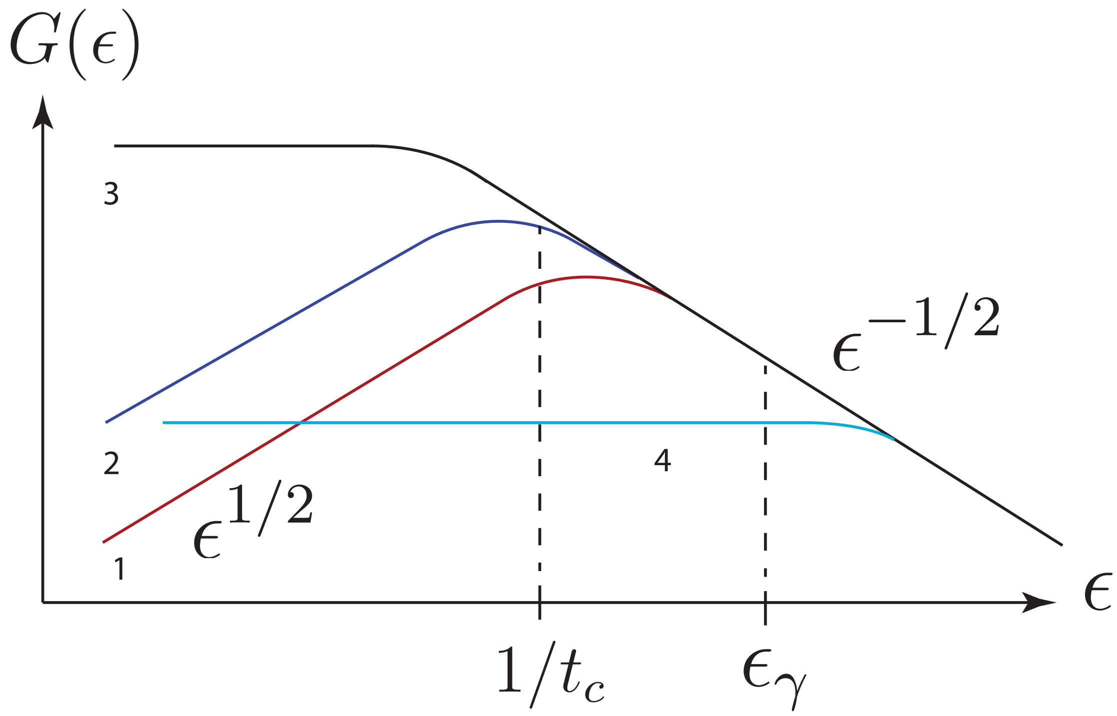

Figure 3-1: Sketch of the Green function $G(\epsilon)$ in log-log scale, in the limit of $N \gg 1$ for several values of $b = \gamma N$ ordered as $b_1 < b_2 < b_c < b_3 < b_4$, with critical b_c of the order unity. NFL-FL transition occurs between blue (2) and grey (3) lines. Light-blue line (4) corresponds to large $b \geq \sqrt{N}$ when 4-fermion interaction is relevant at high energies above $\epsilon_\gamma \gg 1/t_c$ only.

Chapter 4

Perturbed Sachdev-Ye-Kitaev model: a polaron in the hyperbolic plane.

The material for this part of the thesis is taken from our paper [24].

4.1 Introduction

In the present chapter, we extend the results of the previous chapter and works [2, 3] in two important directions. First, we present a method to analyse the $\mathrm{SYK}_4+\mathrm{SYK}_2$ model beyond perturbation theory or its RG-like variant. Second, our new method works at a finite temperature. In particular, we study a previously unexplored range of intermediate strengths of the SYK_2 term, $J/N \ll \Gamma \ll J/\sqrt{N}$. In this regime, the quadratic perturbation is weak at the time scale $t \le t_0 \sim N/J$, where the saddle-point (conformal) solution G_{sp} is applicable. Paradoxically, we find that the perturbation <u>stabilizes</u> the conformal solution $G_{sp}(t) \sim 1/\sqrt{t}$ for extended times, $t \gg t_0$, where the Green function of the pure SYK_4 model is modified by the soft mode fluctuations. Only at the longest time scale, $t \ge J/\Gamma^2 \gg t_0$, the conformal solution G_{sp} gives way to the Fermi-liquid solution $G_{sp}(t) \propto 1/t$.

Our results are best understood using the geometric interpretation [22] of the Schwarzian theory in terms of an auxiliary particle whose trajectories are closed curves in the hyperbolic plane. The quadratic perturbation is then described as the particle being coupled to a free scalar Bose field. For sufficiently strong coupling,

$\Gamma \gg J/N$, a polaron-type bound state is formed, resulting in increased rigidity of the curve and the suppression of its fluctuations.

4.2 Geometric interpretation and the polaron analogy.

The following picture is conceptually important, but we will do most calculations by a different method. Therefore, this section will be brief; more details can be found in the Appendix B. The duality between the Schwarzian action and JT gravity was discussed in Ref. [22] as well as in the papers [19, 27, 13]

The geometric interpretation of the Schwarzian action [22] is based on a correspondence between functions φ as described above and closed curves on the hyperbolic plane. Such curves may be parametrized by the proper length $\ell = J\tau$, which will be used instead of τ for the purpose of this discussion. In the Poincare disk model with metric $ds^2 = \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\varphi^2)$, the curve is given by the equations $\varphi = \varphi(\ell)$ and $r = 1 - \varphi'(\ell)$. This representation is valid if $\varphi''(\ell) \ll \varphi'(\ell) \ll 1$, which is true for a typical curve of length $L = J\beta \gg 1$ in the statistical ensemble. Under the same conditions, we have $Sch(e^{i\varphi(\ell)}, \ell) = K - 1$, where K is the extrinsic curvature of the curve at the given point. This allows for an elegant representation of the Schwarzian action $S_{Sch} = -\alpha_s N \int_0^L Sch(e^{i\varphi(\ell)}, \ell)\, d\ell$ (here $L = \beta J$) in terms of the length of the curve and the enclosed area; however, some regularization is necessary in order to define the functional integral [22]. Replacing the function φ with the curve X, we may rewrite Eq. (3.7) as follows:

$$S[X] = S_{Sch}[X] - \frac{N\Gamma^2}{4J^2} b^{2\Delta} \int G_\Phi^2\big(X(\ell_1), X(\ell_2)\big)\, d\ell_1\, d\ell_2, \tag{4.1}$$

where $G_\Phi(r_1, \varphi_1; r_2, \varphi_2) \propto |\varphi_{12}|^{-2\Delta}(1-r_1)^\Delta(1-r_2)^\Delta$ near the disk boundary. The function G_Φ can be identified with the propagator of a scalar boson Φ. Thus, the nonlocal interaction between different points of the curve is decoupled, such that the action (4.1) is obtained from

$$S[X,\Phi] = S_{Sch}[X] + S_{\Phi}[\Phi] + \int_0^L \Phi(X(\ell))\, d\ell \tag{4.2}$$

where $S_\Phi = \frac{1}{4g\alpha_s N}\int d\mu \Phi(x)(-\nabla^2 - \frac{1}{4} + \delta^2)\Phi(x)$, by integrating out Φ.

The action (4.2) is similar to the polaron problem, where an electron in a crystal interacts with an elastic deformation. By analogy with the heavy polaron, we will look for a mean-field solution where the field Φ forms a potential well close to the boundary of the Poincare disk. The general form of Φ in this region is $\Phi(r,\varphi) = \Lambda(\varphi)(1-r)^\Delta$, and the solution in question is $\Lambda(\varphi) = const$. The curve roughly follows the circle $r = 1 - \frac{2\pi}{L}$ and slightly wiggles. This behavior may be understood as a localized state of a quantum particle, whose coordinate is conveniently defined as $\xi = -\ln(\alpha_s N(1-r))$.

4.3 Adiabatic action.

We proceed with a formal solution for the polaron. It is convenient to rescale time as $\tau \to \frac{J\tau}{\alpha_s N}$ and to introduce a similarly rescaled inverse temperature β and a new coupling constant g:

$$\beta = \frac{J\beta}{\alpha_S N}, \qquad g = \frac{b^{2\Delta}}{2}\frac{N\Gamma^2}{J^2}(\alpha_S N)^{2-4\Delta} = \frac{\alpha_S N^2}{4\sqrt{\pi}}\frac{\Gamma^2}{J^2}. \tag{4.3}$$

Then the action (3.7) reads:

$$S[\varphi] = -\int_0^\beta Sch(e^{i\varphi(\tau)},\tau)\, d\tau - \frac{g}{2}\int \left(\frac{\varphi_1'\varphi_2'}{\varphi_{12}^2}\right)^{1/2} d\tau_1\, d\tau_2. \tag{4.4}$$

Now we reduce the path integral with this action to some solvable quantum mechanical problem. To implement this idea, we introduce new time-dependent variables $\xi(\tau) = -\ln(\varphi'(\tau))$ and $\Xi(\tau) = [\varphi'(\tau)]^{1/2}$, and the corresponding Lagrange multipliers $\lambda(\tau)$ and $\Lambda(\tau)$. This means inserting $\delta(\varphi' - e^{-\xi}) = \int_{-i\infty}^{+i\infty} \exp\big(\lambda(\varphi' - e^{-\xi})\big)\frac{d\lambda}{2\pi i}$ and $\delta(\Xi - e^{-\xi/2})$ (expressed likewise using Λ) in the functional integral.

Thus, the action takes the form

$$S[\varphi, \xi, \lambda, \Xi, \Lambda] = \int_0^\beta \left(\frac{\xi'^2}{2} - \lambda \left(\varphi' - e^{-\xi} \right) - \frac{1}{2} e^{-2\xi} - \Lambda \left(\Xi - e^{-\xi/2} \right) \right) d\tau \\ - \frac{g}{2} \iint \frac{\Xi(\tau_1) \Xi_2(\tau_2)}{|\varphi_{12}|} \, d\tau_1 \, d\tau_2. \tag{4.5}$$

We assume that $\beta \gg 1$ so that the term $\frac{1}{2} e^{-2\xi}$ is relatively small. It will be neglected in our analysis.

We treat action (4.5) using adiabatic approximation, with ξ being the fast variable. That is, the functional integral of e^{-S} over ξ is performed under the assumption that $\varphi'(\tau)$, $\lambda(\tau)$, $\Xi(\tau)$, and $\Lambda(\tau)$ are constant at a suitable time scale τ_* (to be determined later). The result has the form $e^{-S_{eff}}$, where

$$\begin{aligned} S_{eff}[\varphi, \lambda, \Xi, \Lambda] = & \int_0^\beta \left(E_0(\lambda, \Lambda) - \lambda \, \varphi' - \Lambda \, \Xi \right) d\tau \\ & - \frac{g}{2} \int |\varphi_{12}|^{-1} \Xi(\tau_1) \Xi(\tau_2) \, d\tau_1 \, d\tau_2 \end{aligned} \tag{4.6}$$

and $E_0(\lambda, \Lambda)$ is the ground state of the effective Hamiltonian for the variable ξ,

$$\hat{H}_{\lambda, \Lambda} = -\frac{1}{2} \partial_\xi^2 + \Lambda e^{-\xi/2} + \lambda e^{-\xi}. \tag{4.7}$$

This Hamiltonian has bound states with energies

$$E_n = -\frac{(\kappa - 1 - 2n)^2}{32}, \qquad n = 0, \dots, \left\lfloor \frac{\kappa - 1}{2} \right\rfloor, \tag{4.8}$$

where $\kappa = -\sqrt{\frac{8}{\lambda}} \Lambda$. The corresponding eigenfunctions $\psi_n(\xi)$ are provided in the Appendix B. The characteristic time for the adiabatic approximation can be estimated as $\tau_* \sim (E_1 - E_0)^{-1} = \frac{8}{\kappa - 2}$. Such an estimate is certainly correct for a harmonic oscillator, where the oscillation period is the only relevant time scale. The Hamiltonian (4.7) is similar if $\kappa \gg 1$. We will see that the last condition actually guarantees adiabaticity, i.e. that ϕ', λ, Ξ, Λ do not fluctuate at the time scale τ_*. In fact, the fluctuations at all time scales are small enough to be considered Gaussian. Our next goal is to derive an effective action for φ. To this end, we find the saddle point of the action (4.6) with respect to the other variables. The saddle point conditions for λ, Λ, and Ξ read:

$$\varphi' = \tfrac{\partial E_0}{\partial \lambda} = \tfrac{\kappa-1}{32}\tfrac{\kappa}{\lambda}\,, \qquad \Xi = \tfrac{\partial E_0}{\partial \Lambda} = -\tfrac{\kappa-1}{16}\tfrac{\kappa}{\Lambda}\,, \tag{4.9}$$

$$\Lambda(\tau_1) = -g \int d\tau_2\, \tfrac{\Xi(\tau_2)}{|\varphi_{12}|}\,. \tag{4.10}$$

Eqs. (4.9) allow one to eliminate λ and Λ from various formulas; in particular, the definition of κ is equivalent to the relation $\Xi^2 = \frac{\kappa-1}{\kappa}\varphi'$. The integrand in the first term of the action (4.6) can be written as

$$E_0(\lambda,\Lambda) - \lambda\,\varphi' - \Lambda\,\Xi = \frac{\kappa-1}{32}, \tag{4.11}$$

and Eq. (4.10) becomes an equation for $\kappa(\tau)$:

$$\kappa^2(\tau_1)\,\eta(\tau_1) = 16g \int d\tau_2\, \frac{\eta(\tau_2)\sqrt{\varphi'(\tau_1)\varphi'(\tau_2)}}{|\varphi_{12}|}, \tag{4.12}$$

where $\eta(\tau) = \sqrt{1-\kappa^{-1}(\tau)}$. Finally, the effective action is reduced to

$$S = \int_0^\beta \frac{\kappa-1}{32} d\tau - \frac{g}{2}\int d\tau_1 d\tau_2 \frac{\eta(\tau_1)\eta(\tau_2)\sqrt{\varphi'(\tau_1)\varphi'(\tau_2)}}{|\varphi_{12}|}. \tag{4.13}$$

Now, let $\kappa \gg 1$ so that $\eta(\tau) \approx 1$. Furthermore, we will assume (and later verify) that the fluctuations are small, and hence, both $\varphi' \approx 2\pi/\beta$ and κ are nearly constant. Then Eq. (4.12) is simplified as follows:

$$\kappa^2 = 16g \int \frac{d\varphi(\tau_2)}{\left|2\sin\left(\frac{\varphi(\tau_1)-\varphi(\tau_2)}{2}\right)\right|} \approx 32g \ln\left(\frac{\kappa\beta}{16\pi}\right), \tag{4.14}$$

where we have used the cutoff $|\tau_1 - \tau_2| > \tau_* \approx \frac{8}{\kappa}$ for the logarithmic integral. As for the effective action (4.13), its first term may be neglected (see Appendix B). Expressing φ' as a function of φ, namely, $\varphi'(\tau) = \varepsilon(\varphi)$, we get:

$$S \approx -\frac{g}{2}\int_0^{2\pi}\int_0^{2\pi} \frac{d\varphi_1}{\varepsilon(\varphi_1)}\frac{d\varphi_2}{\varepsilon(\varphi_2)} \left(\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^2}\right)^{1/2}. \tag{4.15}$$

4.4 Saddle point solution.

The action (4.15) attains its minimum at the constant field configuration, $\varepsilon(\varphi) = 2\pi/\beta$, as well as all configurations related to it by $PSL(2,R)$ symmetries. The minimum value of the action, $S_{\min} \approx -\beta\kappa^2/32$, determines a correction to the SYK free energy: $F \approx E_0 - Ns_0 T - \frac{J}{\alpha_s N}\frac{\kappa^2}{32}$. Differentiating it and using Eq.(4.14), we find the entropy of the system:

$$S(T) \approx Ns_0 - \frac{gJ}{\alpha_s NT}. \tag{4.16}$$

Note that the entropy vanishes at $T \sim \Gamma^2/J$, which is roughly the temperature at which the SYK_4 conformal Green function $G_{sp}(\tau,0) \sim -(J\tau)^{-1/2}$ gives way to the Fermi liquid behavior, $G(\tau,0) \sim -(\Gamma\tau)^{-1}$. At lower temperatures, $T \leq \Gamma^2/J$, we need to modify our polaron solution. Namely, we should introduce an upper cutoff in Eq. (4.14), $|\tau_1 - \tau_2| < J/\Gamma^2$. Thus, the equation for κ becomes

$$\kappa^2 = 32g\ln\left(\frac{\kappa\overline{\beta}}{16\pi}\right), \qquad \overline{\beta} = \frac{J}{\alpha_s N}\min\left(\beta, \frac{J}{\Gamma^2}\right). \tag{4.17}$$

Fluctuations. Let us estimate the fluctuation of $\varepsilon(\varphi)$ around $\varepsilon_0 = 2\pi/\beta$ and show that they are small. We assume that $T \gg \Gamma^2/J$. In addition to the adiabatic action S given by Eq. (4.15), we need the first non-adiabatic correction. The latter is identical to the Schwarzian action (see Appendix B). We consider the Fourier series $\varepsilon(\varphi) = \varepsilon_0 + \frac{1}{2\pi}\sum_n \delta\varepsilon_n e^{in\varphi}$, expand the effective action $S_{Sch} + S$ up to the second order in $\delta\varepsilon_n$ with $n \neq 0$, and calculate the Gaussian expectation values $\langle\delta\varepsilon_n\delta\varepsilon_{-n}\rangle$. (Note that $\delta\varepsilon_0$ is determined by the equation $\int \frac{d\varphi}{\varepsilon(\varphi)} = \beta$.) This calculation, which can be found in the Appendix, gives the following result:

$$K_\varepsilon(n) \equiv \frac{\langle\delta\varepsilon_n\delta\varepsilon_{-n}\rangle}{\varepsilon_0^2} = \frac{2\pi\varepsilon_0}{\varepsilon_0^2(n^2-1) + (g/2)\tilde{\psi}(n)} \tag{4.18}$$

where $\tilde{\psi}(n) = \psi(n+\frac{1}{2}) - \psi(-\frac{1}{2})$ and $\psi(x)$ is the digamma function; thus, $\tilde{\psi}(n) \approx \ln(n)$ for $n \gg 1$. Eq. (4.18) is accurate for $n \ll n_* \equiv (\varepsilon_0\tau_*)^{-1} = \kappa\beta/(16\pi)$ because it was derived from the effective action that is valid at sufficiently long times, $\tau \gg$

$\tau_* = 8/\kappa$. However, no inconsistency occurs at greater values of n: for $n \gtrsim n_*$, the first term in the denominator of (4.18) starts to dominate over the second one, and the fluctuations are suppressed. The summation of the r.h.s. of Eq. (4.18) over all n, or just those with $|n| \leq n_*$, leads to the estimate

$$\frac{\langle(\delta\varepsilon)^2\rangle}{\varepsilon_0^2} \approx \frac{1}{\epsilon_0 n_*} = \frac{8}{\kappa} \ll 1. \tag{4.19}$$

Thus, the fluctuations of $\varepsilon(\varphi)$ are much smaller than its typical value if $\kappa \gg 8$.

4.5 Phase diagram.

As parameter κ decreases toward unity, the fluctuations become strong and the adiabatic approximation breaks down. At $\kappa \sim 1$, we expect a transition into another phase of our model, where the SYK_2 term is irrelevant at all time scales [25]. In terms of the original parameters of the model, the transition occurs when Γ becomes smaller than its critical value given by the equations

$$\Gamma_c \sim \frac{J}{\sqrt{\alpha_s N^2 \ln \frac{\overline{\beta}}{16\pi}}}, \qquad \overline{\beta} = \frac{J}{\alpha_s N} \min\left(\frac{1}{T}, \frac{J}{\Gamma_c^2}\right), \tag{4.20}$$

Note that the critical value $\Gamma_c = \Gamma_c(T)$ decreases logarithmically with the decrease of the physical temperature T while $T \gtrsim \frac{\Gamma_c^2(T)}{J}$. At lower temperatures, $\Gamma_c(T)$ remains constant. The lines $\Gamma = \Gamma_c(T)$ and $T = J/(\alpha_s N)$ (see Figure 4-1) separate the region with strong fluctuations, characterized by the Green function $G_{fluc}(\tau, 0) \sim \alpha_s N (J\tau)^{-3/2}$ for sufficiently large τ (but still much less than β), from regions where the saddle point solution is valid.

4.6 Higher-order Green functions.

Conformal solution (2.15) describes single-particle fermion Green function of the original problem with Hamiltonian (2.1). Additional information on its quantum dynamics is provided by higher-order fermion Green functions defined as $G^{(p)}(\tau, \tau') \equiv \left(-\frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau')\right)^p$. It can be shown (see Appendix B, sec.III) that the functions

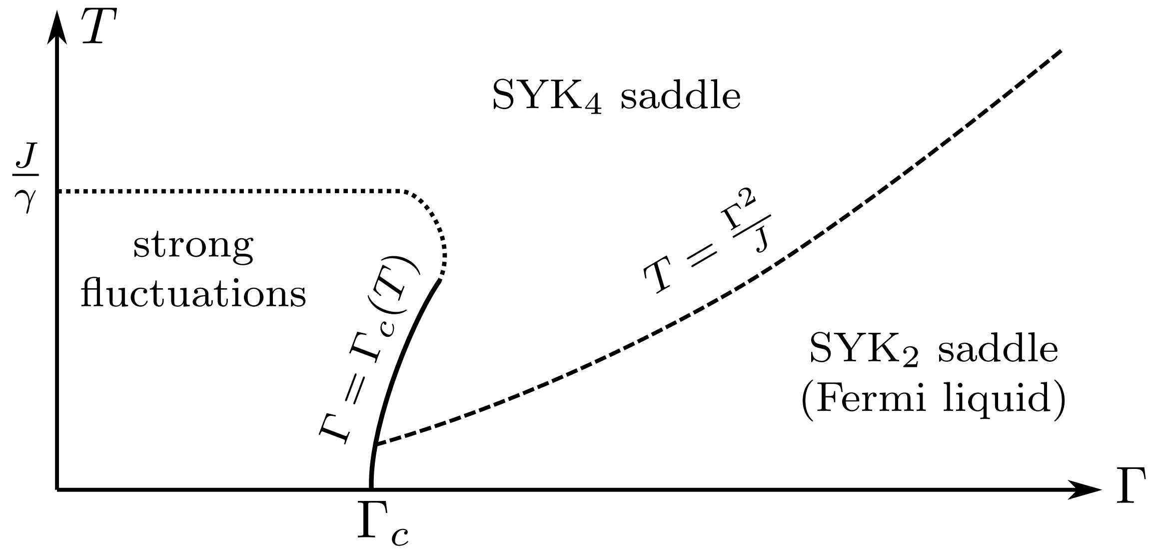

Figure 4-1: Regions with different Green function behaviors at large τ. The phase boundaries are not sharp, except for the boundary between the fluctuation region and saddle point regions (the solid line). The latter is well-defined asymptotically, under the condition $T \ll J/(\alpha_s N)$. Hence, the termination point of the solid line, given by the condition $T \sim J/(\alpha_s N)$, is fuzzy.

$G^{(p)}(\tau,\tau')$ with $p \ll \kappa$ can be calculated by means of the effective action (4.15) and its propagator (4.18). To find them, we need just to average p-power of the conformal solution (2.15) over fluctuations of variables ξ and φ described by the polaron bound-state: $G^{(p)}(\tau_1,\tau_2) = (-1)^p \langle \left[b e^{-\xi_1-\xi_2} \sin^{-2}(\frac{1}{2}(\varphi_1-\varphi_2))\right]^{p/4}\rangle$. The result of calculations (provided in the Appendix B, sec.III) reads (remember that $\kappa \ll N$):

$$\frac{G^{(p)}(\tau_1-\tau_2)}{[G(\tau_1-\tau_2)]^p} = \exp\left[\frac{p^2}{4\kappa}\left(1+f(\theta_{12})\right)\right] \tag{4.21}$$

where $\theta_{12} = 2\pi T(\tau_1-\tau_2)$ and function $f(\theta)$ is provided below ($n_*\varepsilon_0 = \kappa/8$):

$$\begin{aligned} f(\theta) &= \tfrac{2+n_*\theta}{(n_*\theta)^2}\left[2n_*\theta\cosh\left(\tfrac{n_*\theta}{2}\right) - 4\sinh\left(\tfrac{n_*\theta}{2}\right)\right]\exp\left\{-\tfrac{n_*\theta}{2}\right\} = \\ &= \begin{cases} 1 & n_*\theta \gg 1 \\ \frac{\theta n_*}{3} & n_*\theta \ll 1 \end{cases} \end{aligned} \tag{4.22}$$

Two terms in the exponent of Eq.(4.21) come from the averaging over fluctuations of $\xi_{1,2}$ (1st term) and angular variables $\varphi_{1,2}$.

4.7 Conclusion

We have shown that a moderate quadratic perturbation to the SYK_4 model with N Majorana modes can be described in terms of a self-consistent polaron-type solution. The presence of such a perturbation with strength Γ in the interval $J/N \leq \Gamma \leq J/\sqrt{N}$ stabilizes conformal saddle-point solution for the Majorana Green function within a broad range of energies and temperatures. The SYK_4 mean-field Green function $G(\epsilon) \sim 1/\sqrt{J\epsilon}$ (defined at Matsubara frequencies $\epsilon = i \cdot 2\pi T\big(n + \frac{1}{2}\big)$) is valid down to $T_* \sim \Gamma^2/J$, where a crossover to a Fermi liquid at $\varepsilon \lesssim T_*$ occurs. Schematic "phase diagram" of the model is shown in 4-1. At low temperatures, there is a genuine phase transition at $\Gamma = \Gamma_c$, where Γ_c is defined in (4.20). Specifically at $T = 0$, the $\epsilon \to +0$ asymptotic changes from $G(\epsilon) \propto \sqrt{\epsilon}$ for $\Gamma < \Gamma_c$ to $G(\varepsilon) \simeq 1/\Gamma$ for $\Gamma > \Gamma_c$. Note that higher-order Green functions $G^{(p)}(\tau)$ display exponential growth with p, Eq.(4.21).

Chapter 5

Conclusion

In conclusion, we would like to repeat the problem we investigated and the result. We have considered the SYK model, where interactions play a crucial role due to the degeneration of the one-particle energy levels. This model has two qualitatively different regimes: for short times, the model is described by the mean-field equation, and we need to take fluctuations into account for long times. Both regimes demonstrate non-Fermi-liquid behavior.

We have studied the influence of the perturbation which lifts the degeneracy. The perturbation is important on the mean-field level, but we need to take into account strong fluctuations for weak perturbations. These fluctuations can be described by the Hamiltonian of the Liouville quantum mechanics. This Hamiltonian does not have bound states, and it is the origin of strong fluctuations. Our analysis showed that in the limit $\Gamma \to 0$, where Γ is a typical scale of perturbation, perturbation does not change the asymptotic behaviour. For a particular value of perturbation strength ($\Gamma = \Gamma_c$), the bound state will appear in the Hamiltonian, and fluctuations will be suppressed. According to this scenario, we have plotted a phase diagram of our system: 4-1. This diagram is the main result of the thesis.

The more the amplitude of perturbations, the more bound states in the effective Hamiltonian. We have shown that the fluctuation correction to the Green function is suppressed for the system with large numbers of bound states.

The following questions could be investigated connected to our conclusion. Are these results applicable for the description of the extended system such as mentioned in the work [30]? Are energy levels of the bound states of the effective Hamiltonian observable?

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Appendix A

Calculation of the correction to the Green function in the presence of the quadratic perturbation.

A.1 Plan of calculation

We will present method for evaluation of various correlation function for general SYK_q model with q simultenously interacting fermions; then we will put $q = 4$ in the end of the calculations. The low-energy limit of SYK_q model is described by the "sigma-model" action over the manifold of monotonic functions $f(\tau)$, which corresponds to the re-parametrization symmetry of the mean-field solution in the scaling limit. The functions defined on this manifold can be conveniently parametrised in terms of the field $\phi(\tau)$, which is defined according to $f'(\tau) = e^{\phi(\tau)}$. In such a representation, the action reduces to the simple form

$$S = -\frac{M}{2}\int (\phi')^2 d\tau, \tag{A.1}$$

where parameter M depends on the number of fermions N and on the value of q. In such a parametrization the measure of the functional integration is flat. Note that here and below in the Appendix A we put interaction strength $J = 1$. The field $G(\tau, \tau')$, which becomes equal to the fermionic Green function upon integrating over

$\phi(\tau)$, reads in the parametrization as follows:

$$G(\tau,\tau') = sign(\tau-\tau')b^{\Delta}\frac{e^{\Delta\phi(\tau)}e^{\Delta\phi(\tau')}}{|\int_{\tau'}^{\tau} e^{\phi(\tilde{\tau})}d\tilde{\tau}|^{2\Delta}} \tag{A.2}$$

with $\Delta = \frac{1}{q}$ and $b = (\frac{1}{2}-\Delta)\frac{\tan(\pi\Delta)}{\pi}$. To simplify the notation in the following, we will consider averaging of the objects like $G^{\frac{n}{\Delta}}$. The average Green function will come up as a specific result at $n = \Delta$.

Below in Sec. II we re-derive some results from Ref.[[4]] in a slightly different way; namely, we show how to reduce evaluation of the Green function (A.2) with the action (A.1) to the calculation of matrix elements of the Liouville quantum mechanics.

In Sec. III we evaluate correlation functions of the products of various powers of Green functions $G^{\frac{n}{\Delta}}$ which are necessary to calculate the corrections to the Green function generated by the perturbation of the SYK_2 type:

$$\delta\langle G^{n/\Delta}_{\tau_1,\tau_2}\rangle = -\langle G^{n/\Delta}_{\tau_1,\tau_2} S_{int}\rangle + \langle G^{n/\Delta}_{\tau_1,\tau_2}\rangle\langle S_{int}\rangle. \tag{A.3}$$

$$S_{int} = -\frac{\Delta N\Gamma^2}{2m}\int d\tau_1 d\tau_2 G^{m/\Delta}(\tau_1,\tau_2). \tag{A.4}$$

Finally we will be interested in the case $m = \frac{1}{2}$ and $n = \frac{1}{4}$, $\Delta = \frac{1}{4}$.

To simplify further formulae, we switch to dimensionless time units $t = \frac{\tau}{2M}$ and introduce new notation for the Green function:

$$g_n(t,t') = \left(\frac{1}{\Gamma(2n)}\frac{b^n}{(2M)^{2n}}\right)^{-1} G^{n/\Delta}(\tau,\tau'). \tag{A.5}$$

Therefore Eq.(A.3) with S_{int} from Eq.(A.4) can now be rewritten in the form

$$\delta\langle G^{n/\Delta}_{\tau_1,\tau_2}\rangle = \frac{N\Gamma^2\Delta}{m}\frac{b^{(n+m)}}{\Gamma(2n)\Gamma(2m)(2M)^{2(n+m)-2}}\times$$
$$\int_{t_3>t_4} dt_3 dt_4 \left[\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle - \langle g_n(t_1,t_2)\rangle\langle g_m(t_3,t_4)\rangle\right]. \tag{A.6}$$

To evaluate expression (A.6) one has to calculate the average $\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle$.

The difficulty of this calculation is due to the absence of Wick contraction rules, so all possible time orderings have to be considered explicitly. Symmetry of Green function allows us to fix relations $t_1 > t_2$ and $t_3 > t_4$, leaving 6 possible time orderings: 1) $t_1 > t_2 > t_3 > t_4$, 2) $t_1 > t_3 > t_2 > t_4$, 3)$t_3 > t_1 > t_2 > t_4$, 4) $t_1 > t_3 > t_4 > t_2$, 5) $t_3 > t_1 > t_4 > t_2$ and 6) $t_3 > t_4 > t_1 > t_2$. The orderings 1 and 6 are trivial [4]:

$$\langle g_n(t_1, t_2) g_m(t_3, t_4)\rangle = \langle g_n(t_1, t_2)\rangle\langle g_m(t_3, t_4)\rangle. \tag{A.7}$$

In Sec. III we present evaluation of the average values corresponding to the remaining four variants of time ordering. In the remaining Secs. IV - VI we combine various contributions in the long-time limit and derive the final result.

A.2 Averaging various powers of the Green function

First of all, we re-derive some results from Ref. [4] in a slightly different way. We start from the formula for average power of the Green function

$$\langle G^{n/\Delta}(\tau, \tau')\rangle = \int D\phi\ b^{n\Delta} \frac{e^{n\phi(\tau)} e^{n\phi(\tau')}}{|\int_{\tau'}^{\tau} e^{\phi(\tilde{\tau})} d\tilde{\tau}|^{2n}} e^{-\frac{M}{2}\int (\phi')^2 d\tau}. \tag{A.8}$$

Switching to dimensionless time, we write

$$\langle G^{n/\Delta}(t, t')\rangle = \int D\phi\ \frac{b^n}{(2M)^{2n}} \frac{e^{n\phi(t)} e^{n\phi(t')}}{|\int_{t'}^{t} e^{\phi(\tilde{t})} d\tilde{t}|^{2n}} e^{-\frac{1}{4}\int (\phi')^2 dt}. \tag{A.9}$$

Using identity $\frac{1}{a^{2n}} = \int_0^\infty \frac{\alpha^{n-1}}{\Gamma(2n)} e^{-\alpha a} d\alpha$ one can rewrite above expression as follows:

$$\langle G^n(t, t')\rangle = \int_0^\infty \frac{\alpha^{2n-1} d\alpha}{\Gamma(2n)} \int D\phi\ \frac{b^n}{(2M)^{2n}} e^{n\phi(t)} e^{n\phi(t')} e^{-\frac{1}{4}\int (\phi')^2 dt - \alpha \int_{t'}^{t} e^{\phi(\tilde{t})} d\tilde{t}}. \tag{A.10}$$

Functional integral over $\phi(t)$ can be interpreted as a quantum mechanical amplitude and evaluated explicitly. There is a technical problem however: the field $\phi(t)$ in Eq. (A.9) can be shifted by a constant: $\phi(t) \to \phi(t) + \phi_0$, producing a divergent integral. In the calculation provided in Ref. [4], this zero mode appeared as an infinite multiplicative constant, coming from divergent integration over parameter α. This divergence was argued [4] to be irrelevant since it is related to the symmetry of the action. Slightly different formulation of the same approach is to put α equal to 1 instead of integration over α. Here we first check this idea by using another method. Namely, we fix the "gauge condition" by putting the value of $\phi(t)$ equal to ϕ_0 and then integrate over α; we obtain then the same result as in Ref.[4]. Therefore in our further calculations we will follow the approach of Ref. [4] which is simpler in implementation.

Rewriting formulae in terms of $g_n(t,t')$, we find

$$\langle g_n(t,t')\rangle = \int_0^\infty \alpha^{2n-1} d\alpha \int d\phi_1 \langle \phi_0 | e^{n\phi} U_\alpha(t,t') e^{n\phi} | \phi_1 \rangle, \tag{A.11}$$

where $U_\alpha(\tau,\tau')$ is the evolution operator corresponding to the Liouville's Hamiltonian $H = -\frac{1}{4}\partial_\phi^2 + \alpha e^\phi$. It can be written as

$$U_\alpha(t,t') = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')} |k,\alpha\rangle\langle k,\alpha| \tag{A.12}$$

with eigenstates

$$\langle \phi | k, \alpha \rangle = \frac{2}{\Gamma(2ik)} K_{2ik}(2\sqrt{\alpha e^\phi}). \tag{A.13}$$

It is more convenient to work with Mellin transformed eigenfunctions

$$\frac{2}{\Gamma(2ik)} K_{2ik}(2\sqrt{x}) = \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(p-ik)\Gamma(p+ik)}{\Gamma(2ik)} x^{-p} \frac{dp}{2\pi i}. \tag{A.14}$$

We now introduce the "matrix element" $G(p,k)$ as follows:

$$G(p,k) = \frac{\Gamma(p-ik)\Gamma(p+ik)}{\Gamma(2ik)}. \tag{A.15}$$

Then Eq. (A.11) becomes

$$\langle g_n(t,t')\rangle = \int_0^\infty \tfrac{dk}{2\pi} e^{-k^2(t-t')} \int_0^\infty \alpha^{2n-1} d\alpha \int_{-\infty}^\infty d\phi_1 e^{n\phi_0} e^{n\phi_1} \langle \phi_0|k,\alpha\rangle\langle k,\alpha|\phi_1\rangle =$$
$$= \int_0^\infty \tfrac{dk}{2\pi} e^{-k^2(t-t')} \int_0^\infty \alpha^{2n-1} d\alpha \int_{-\infty}^\infty d\phi_1 e^{n\phi_0} e^{n\phi_1} \int \tfrac{dp_1}{2\pi i}\tfrac{dp_1}{2\pi i} \times$$
$$G(p_1,k)G(p_1,-k)\alpha^{-p_1}e^{-p_1\phi_0}\alpha^{-p_2}e^{-p_2\phi_1} =$$
$$= \int_0^\infty \tfrac{dk}{2\pi} e^{-k^2(t-t')} G(n,k)G(n,-k). \qquad \text{(A.16)}$$

In the limit of $t \gg 1$ we find

$$\langle g_n(t,t')\rangle = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')}\Gamma^4(\Delta n)(4k^2) = \frac{\Gamma^4(n)}{2\sqrt{\pi}(t-t')^{\frac{3}{2}}}. \qquad \text{(A.17)}$$

which coinsides with the result of Ref. [4].

A.3 Averaging the products of various powers of Green function.

We now turn to the calculation of the averages of the type $\langle g^n(t_1,t_2)g^m(t_3,t_4)\rangle$. Following the same steps as in the Sec. II we obtain

$$\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle = \int_0^\infty \alpha^{2n-1} d\alpha \int_0^\infty \beta^{2m-1} d\beta \int D\phi e^{n\phi_1} e^{n\phi_2} e^{m\phi_3} e^{m\phi_4} \times$$
$$e^{-\frac{1}{4}\int(\phi')^2 dt - \alpha\int_{t_2}^{t_1} e^{\phi(\tilde{t})}d\tilde{t} - \beta\int_{t_4}^{t_3} e^{\phi(\tilde{t})}d\tilde{t}}, \text{(A.18)}$$

with shorthand notation $\phi_i = \phi(t_i)$. Like in Sec. II, we interpreter the functional integral over $\phi(t)$ as a quantum-mechanical amplitude. It is convenient to fix the "gauge" by setting $\alpha \to 1$, to simplify calculations. The result of averaging depends crucially on the specific time ordering (see discussion in Sec. I). We present here details of the calculation for the cases 2 and 3. Results for the cases 4 and 5 can be obtained in similar way, so we will provide the results only.

A.3.1 Time-ordering 2: $t_1 > t_3 > t_2 > t_4$

Quantum mechanical representation of the problem corresponds to the free particle motion at times $t < t_4$. In the range of times $t_4 < t < t_2$ the exponential potential e^{ϕ} with the magnitude equal to β is turned on, so the evolution during this time interval is described by $U_\beta(t_2, t_4)$ (see Sec. II for the definition of $U(t, t')$). Next, in the time region between t_2 and t_3, the evolution is governed by $U_{1+\beta}(t_3, t_2)$, and between t_1 and t_3 it is given by $U_1(t_1, t_3)$. Finally, at $t > t_1$ the particle is free again.

Than quantum-mechanical average is of the following form (hereafter we use Roman subscripts to denote specific time ordering, which is the 2nd one currently):

$$\begin{aligned}\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle_{II} = \int_0^\infty \beta^{2m-1} d\beta \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{n\phi_1} e^{n\phi_2} e^{m\phi_3} e^{m\phi_4} \\ \langle\phi_1|U_1(t_1,t_3)|\phi_3\rangle\langle\phi_3|U_{1+\beta}(t_3,t_2)|\phi_2\rangle\langle\phi_2|U_\beta(t_2,t_4)|\phi_4\rangle.\end{aligned} \tag{A.19}$$

Using explicit representation for U we find

$$\begin{aligned}\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle_{II} = \int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{1,3}} e^{-k_2^2 t_{3,2}} e^{-k_3^2 t_{2,4}} \int_0^\infty \beta^{2m-1} d\beta \\ \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{n\phi_1} e^{n\phi_2} e^{m\phi_3} e^{m\phi_4} \langle\phi_1|k_1,1\rangle \times \\ \langle k_1,1|\phi_3\rangle\langle\phi_3|k_2,1+\beta\rangle\langle k_2,1+\beta|\phi_2\rangle\langle\phi_2|k_3,\beta\rangle\langle k_3,\beta|\phi_4\rangle.\end{aligned} \tag{A.20}$$

The integration over ϕ are factorized. Integration over ϕ_1 and ϕ_4 are trivial:

$$\int d\phi_1 e^{n\phi_1}\langle\phi_1|k_1,1\rangle = \int d\phi_1 e^{n\phi_1} \int_{c-i\infty}^{c+i\infty} G(p,k_1) e^{-p\phi_1} \frac{dp}{2\pi i} = G(n,k_1). \tag{A.21}$$

$$\begin{aligned}\textstyle\int d\phi_4 e^{m\phi_4}\langle k_3,\beta|\phi_4\rangle = \int d\phi_4 e^{m\phi_4} \int_{c-i\infty}^{c+i\infty} G(p,-k_3) e^{-\phi_4 p}\beta^{-p}\frac{dp}{2\pi i} \\ = G(m,-k_3)\beta^{-m}.\end{aligned} \tag{A.22}$$

Integration over ϕ_3 and ϕ_2 are more involved. Integrating over ϕ_3 we find

$$\begin{gathered}
\int d\phi_3 e^{m\phi_3}\beta^m \langle k_1, 1|\phi_3\rangle\langle\phi_3|k_2, 1+\beta\rangle = \\
\int d\phi_3 e^{m\phi_3}\beta^m \int_{c-i\infty}^{c+i\infty} G(p_1, -k_1)G(p_2, k_2)e^{-\phi_3(p_2+p_1)}(1+\beta)^{-p_2}\tfrac{dp_1 dp_2}{(2\pi i)^2} = \\
\beta^m \int_{c-i\infty}^{c+i\infty} G(m-p, -k_1)G(p, k_2)(1+\beta)^{-p}\tfrac{dp}{2\pi i} = \\
(1+\beta)^{ik_2}\beta^m \tfrac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(m+ik_2, k_1)G(m-ik_2, -k_1)\boldsymbol{F}(m+ik_2-ik_1, m+ik_2+ik_1, 2m, -\beta) = \\
(1+\beta)^{ik_2}\tfrac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(m-ik_2, -k_1)\int_{c_m-i\infty}^{c_m+i\infty} G(p+ik_2, k_1)\tfrac{\Gamma(m-p)}{\Gamma(m+p)}\beta^p \tfrac{dp}{2\pi i}
\end{gathered} \tag{A.23}$$

with $\boldsymbol{F}(a,b,c,z)$ for the normalized Hyperheometric function, $c>0$ and $c_m \in (0, m)$. Integration over ϕ_2 can be performed in the similar manner. We note a useful identity:

$$\begin{gathered}
\int d\phi \beta^m e^{m\phi}\langle k_L, \alpha|\phi\rangle\langle\phi|k_R, \alpha+\beta\rangle = \int d\phi\beta^m e^{m\phi}\langle -k_R, \alpha+\beta|\phi\rangle\langle\phi| -k_L, \alpha\rangle \\
(1+\frac{\beta}{\alpha})^{ik_R}\frac{\Gamma(2ik_L)}{\Gamma(2ik_R)} G(m-ik_R, -k_L)\int_{c_m-i\infty}^{c_m+i\infty} G(p+ik_R, k_L)\frac{\Gamma(m-p)}{\Gamma(m+p)}\left(\frac{\beta}{\alpha}\right)^p \frac{dp}{2\pi i}.
\end{gathered} \tag{A.24}$$

With this identity, we find

$$\begin{aligned}
\langle g_n(t_1,t_2) g_m(t_3,t_4)\rangle_{II} = \int_0^\infty & \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{1,3}} e^{-k_2^2 t_{3,2}} e^{-k_3^2 t_{2,4}} \int_0^\infty \beta^{-1} G(n, k_1)G(m, -k_3) d\beta \\
& (1+\beta)^{ik_2}\frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(m-ik_2, -k_1)\int_{c_m-i\infty}^{c_m+i\infty} G(p+ik_2, k_1)\frac{\Gamma(m-p)}{\Gamma(m+p)}\beta^p \frac{dp}{2\pi i} \\
& \beta^{ik_2}(1+\beta)^{-ik_2}\frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} G(n+ik_2, k_3)\int_{c_n-i\infty}^{c_n+i\infty} G(q-ik_2, -k_3)\frac{\Gamma(n-q)}{\Gamma(n+q)}\beta^{-q}\frac{dq}{2\pi i}.
\end{aligned} \tag{A.25}$$

The last integration over β gives:

$$\begin{gathered}
\langle g_n(t_1,t_2) g_m(t_3,t_4)\rangle_{II} = \int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{1,3}} e^{-k_2^2 t_{3,2}} e^{-k_3^2 t_{2,4}} \\
G(n, k_1)G(m, -k_3)\frac{\Gamma(2ik_1)}{\Gamma(2ik_2)}\frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} G(n+ik_2, k_3)G(m-ik_2, -k_1) \\
\int_{c_{\min(m,n)}-i\infty}^{c_{\min(m,n)}+i\infty} G(p+ik_2, k_1)\frac{\Gamma(m-p)}{\Gamma(m+p)} G(p, -k_3)\frac{\Gamma(n-p-ik_2)}{\Gamma(n+p+ik_2)}\frac{dp}{2\pi i}.
\end{gathered} \tag{A.26}$$

A.3.2 Time-ordering 3: $t_3 > t_1 > t_2 > t_4$

Following the same steps as above we come to

$$\langle g_n(t_1,t_2) g_m(t_3,t_4)\rangle_{III} = \int_0^\infty \beta^{2m-1} d\beta \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{n\phi_1} e^{n\phi_2} e^{m\phi_3} e^{m\phi_4}$$
$$\langle \phi_3 | U_\beta(t_3,t_1) | \phi_1\rangle \langle \phi_1 | U_{1+\beta}(t_1,t_2)|\phi_2\rangle \langle \phi_2 | U_\beta(t_2,t_4)|\phi_4\rangle. \quad \text{(A.27)}$$

With explicit expression for U, we find

$$\langle g_n(t_1,t_2) g_m(t_3,t_4)\rangle_{III} = \int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{3,1}} e^{-k_2^2 t_{1,2}} e^{-k_3^2 t_{2,4}} \int_0^\infty \beta^{2m-1} d\beta$$
$$\int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{n\phi_1} e^{n\phi_2} e^{m\phi_3} e^{m\phi_4} \langle \phi_3 | k_1, \beta\rangle \langle k_1,\beta|\phi_1\rangle \times$$
$$\langle \phi_1 | k_2, 1+\beta\rangle \langle k_2, 1+\beta | \phi_2\rangle \langle \phi_2 | k_3,\beta\rangle \langle k_3,\beta|\phi_4\rangle. \quad \text{(A.28)}$$

Using the identity in Eq. (A.24) we integrate over ϕ:

$$\langle g_n(t_1,t_2) g_m(t_3,t_4)\rangle_{III} = \int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{3,1}} e^{-k_2^2 t_{1,2}} e^{-k_3^2 t_{2,4}} \int_0^\infty \beta^{-1} d\beta G(m,k_1) G(m,-k_3)$$
$$(1+\frac{1}{\beta})^{-ik_2} \frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} G(n+ik_2,k_3) \int_{c_n - i\infty}^{c_n+i\infty} G(q-ik_2,-k_3) \frac{\Gamma(n-q)}{\Gamma(n+q)} \left(\frac{1}{\beta}\right)^q \frac{dq}{2\pi i}$$
$$(1+\frac{1}{\beta})^{ik_2} \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(n-ik_2,-k_1) \int_{c_n - i\infty}^{c_n+i\infty} G(p+ik_2,k_1) \frac{\Gamma(n-p)}{\Gamma(n+p)} \left(\frac{1}{\beta}\right)^p \frac{dp}{2\pi i}. \quad \text{(A.29)}$$

Finally, β-integration gives

$$\langle g_n(t_1,t_2) g_m(t_3,t_4)\rangle_{III} = \int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{3,1}} e^{-k_2^2 t_{1,2}} e^{-k_3^2 t_{2,4}} G(m,k_1) G(m,-k_3)$$
$$\frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(n-ik_2,-k_1) G(n+ik_2,k_3) \int_{c_n - i\infty}^{c_n+i\infty} G(-p-ik_2,-k_3) G(p+ik_2,k_1) \frac{dp}{2\pi i}. \quad \text{(A.30)}$$

In fact, for time-ordering 3 we can go even further and calculate one of the momentum integrals analytically: p-integration gives the momentum conservation law $2\pi\delta(k_1 - k_3)$ and as a result:

$$\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle_{III} = \int_0^\infty \frac{dk_1 dk_2}{(2\pi)^2} e^{-k_1^2 t_{3,1}} e^{-k_2^2 t_{1,2}} e^{-k_1^2 t_{2,4}} \times \\ G(m,k_1)G(m,-k_1)G(n-ik_1,-k_2)G(n+ik_1,k_2). \tag{A.31}$$

A.3.3 Results for time-orderings 4 and 5

For the orderings 4 and 5 we provide only the results:

$$\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle_{IV} = \int_0^\infty \frac{dk_1 dk_2}{(2\pi)^2} e^{-k_1^2 t_{1,3}} e^{-k_2^2 t_{3,4}} e^{-k_1^2 t_{4,2}} \times \\ G(n,k_1)G(n,-k_1)G(m-ik_1,-k_2)G(m+ik_1,k_2). \tag{A.32}$$

$$\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle_{V} = \int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{-k_1^2 t_{3,1}} e^{-k_2^2 t_{1,4}} e^{-k_3^2 t_{4,2}} \\ G(m,k_1)G(n,-k_3)\frac{\Gamma(2ik_1)}{\Gamma(2ik_2)}\frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)}G(m+ik_2,k_3)G(n-ik_2,-k_1) \\ \int_{c_{\min(m,n)}-i\infty}^{c_{\min(m,n)}+i\infty} G(p+ik_2,k_1)\frac{\Gamma(n-p)}{\Gamma(n+p)}G(p,-k_3)\frac{\Gamma(m-p-ik_2)}{\Gamma(m+p+ik_2)}\frac{dp}{2\pi i}. \tag{A.33}$$

A.4 Cancellation of infra-red singularities for 3rd and 4th time orderings

We need to calculate integrals like the one indicated in Eq.(9) of the main text:

$$f(t_1-t_2) = \int_{t_3<t_4} (\langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle - \langle g_n(t_1,t_2)\rangle\langle g_m(t_3,t_4)\rangle). \tag{A.34}$$

We introduce the following notations for the integrands corresponding to different variants of the time ordering:

$$f_i(t_1-t_2) = \int_{T_i} dt_3 dt_4 \langle g_n(t_1,t_2)g_m(t_3,t_4)\rangle_i \\ f_Z = \int_{t_3<t_4,t_2<t_4,t_3<t_1} \langle g_n(t_1,t_2)\rangle\langle g_m(t_3,t_4)\rangle \tag{A.35}$$

Here T_i is the area of integration which satisfies the ith order of times. Using these functions we can write: $f(t) = \sum_{i=II}^{V} f_i(t) - f_Z(t)$. The correction to the Green function can be expressed via $f(t)$ as it is present in Eq.(10) of the main. Note that the functions f_{III}, f_{IV} and f_Z are not well-defined since the integrals in Eq. (A.35) diverge. Fortunately, these divergencies cancel each other. To demonstrate with fact, we write these function explicitly

$$f_{III}(t_1 - t_2) = \delta_{III}\langle g_n(t_1, t_2)\rangle = \int_{-\infty}^{t_2} dt_4 \int_{t_1}^{\infty} dt_3 \langle g_n(t_1, t_2) g_m(t_3, t_4)\rangle_{III} = \int_0^{\infty} \frac{dk_1}{2\pi} \int_0^{\infty} \frac{dk_2}{2\pi} \frac{e^{-k_2^2(t_1-t_2)}}{k_1^4} G(m, k_1) G(n + ik_1, k_2) G(n - ik_1, -k_2) G(m, -k_1) \tag{A.36}$$

and

$$f_{IV}(t_1 - t_2) = \delta_{IV}\langle g_n(t_1, t_2)\rangle = \int_{t_2}^{t_1} dt_3 \int_{t_2}^{t_3} dt_4 \langle g_n(t_1, t_2) g_m(t_3, t_4)\rangle_{IV} = \int_0^{\infty} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{e^{-k_2^2(t_1-t_2)} - e^{-k_1^2(t_1-t_2)}(1 + (k_1^2 - k_2^2)(t_1 - t_2))}{(k_1^2 - k_2^2)^2} G(n, k_1) G(m + ik_1, k_2) G(m - ik_1, -k_2) G(n, -k_1). \tag{A.37}$$

Finally,

$$f_Z(t_1 - t_2) = \delta_Z\langle g_n(t_1, t_2)\rangle = \int_{t_3 > t_4, t_4 < t_1, t_3 > t_2} dt_3 dt_4 \langle g_n(t_1, t_2)\rangle \langle g_m(t_3, t_4)\rangle = \int_0^{\infty} \frac{dk_1}{2\pi} \int_0^{\infty} \frac{dk_2}{2\pi} \frac{e^{-k_1^2(t_1-t_2)}(1 + k_2^2(t_1 - t_2))}{k_2^4} G(n, k_1) G(n, -k_1) G(m, k_2) G(m, -k_2). \tag{A.38}$$

It is convenient to split f_Z in two parts:

$$f_{Z,III}(t) = \int_0^{\infty} \frac{dk_1}{2\pi} \int_0^{\infty} \frac{dk_2}{2\pi} \frac{e^{-k_2^2 t}}{k_1^4} G(n, k_2) G(n, -k_2) G(m, k_1) G(m, -k_1), \tag{A.39}$$

$$f_{Z,IV} = \int_0^{\infty} \frac{dk_1}{2\pi} \int_0^{\infty} \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} t}{k_2^2} G(n, k_1) G(n, -k_1) G(m, k_2) G(m, -k_2). \tag{A.40}$$

The following combinations are free from divergencies upon integration: $\bar{f}_{III}(t) = f_{III}(t) - f_{Z,III}(t)$ and $\bar{f}_{IV}(t) = f_{IV}(t) - f_{Z,IV}(t)$. In the next Section, we evaluate the asymptotic behaviour of the result of this integration.

A.5 Contribution from the regions III and IV.

In this Section we calculate contributions to the Green function correction coming from the time orderings 3 and 4. They can be represented explicitely as some coefficient mutiplying $\langle g_n(t)\rangle$. To calculate it, we find the asymptotic behavior of $\bar{f}_{III}(t)$ and $\bar{f}_{IV}(t)$ in the limit of long time t. We start from $\bar{f}_{III}(t)$. We use here the fact that for $t \gg 1$ one has $k_2 \ll 1$ and $k_1 \sim 1$:

$$\bar{f}_{III}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_2^2 t}}{k_1^4} G(m,k_1)G(m,-k_1) \times \\ (G(n+ik_1,k_2)G(n-ik_1,-k_2) - G(n,k_2)G(n,-k_2)) \approx \\ \frac{\Gamma(n)^4}{2\sqrt{\pi}t^{\frac{3}{2}}} \int_0^\infty \frac{dk_1}{2\pi} \frac{G(m,k_1)G(m,-k_1)(\Gamma^2(n+ik_1)\Gamma^2(n-ik_1) - \Gamma^4(n))}{\Gamma(n)^4 k_1^4} \equiv C_{III}(n,m)\langle g_n(t)\rangle. \tag{A.41}$$

To evaluate the contribution of the 4th time ordering it is convenient to split it into two parts. The first one is

$$f_{IV,1}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_2^2 t} - e^{-k_1^2 t}}{(k_1^2 - k_2^2)^2} G(n,k_1)G(m+ik_1,k_2)G(m-ik_1,-k_2)G(n,-k_1) = \\ \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_2^2 t} - e^{-k_1^2 t}}{(k_1^2 - k_2^2)^2} \\ \frac{\Gamma^2(n+ik_1)\Gamma^2(n-ik_1)\Gamma(m+ik_1+ik_2)\Gamma(m+ik_1-ik_2)\Gamma(m-ik_1+ik_2)\Gamma(m-ik_1-ik_2)}{\Gamma(2ik_1)\Gamma(-2ik_1)\Gamma(-2ik_2)\Gamma(-2ik_2)}. \tag{A.42}$$

We symmetrize it over interchange of $k_{1,2}$:

$$f_{IV,1}(t) = \frac{1}{2} \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_2^2 t} - e^{-k_1^2 t}}{(k_1^2 - k_2^2)^2} \\ \frac{\Gamma^2(n+ik_1)\Gamma^2(n-ik_1) - \Gamma^2(n+ik_2)\Gamma^2(n-ik_2)}{\Gamma(-2ik_1)\Gamma(2ik_1)} G(m+ik_1,k_2)G(m-ik_1,-k_2). \tag{A.43}$$

As a result:

$$f_{IV,1}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_2^2 t}}{(k_1^2 - k_2^2)^2} \frac{\Gamma^2(n+ik_1)\Gamma^2(n-ik_1) - \Gamma^2(n+ik_2)\Gamma^2(n-ik_2)}{\Gamma(-2ik_1)\Gamma(2ik_1)} \times$$
$$G(m+ik_1,k_2)G(m-ik_1,-k_2) \approx$$
$$\frac{\Gamma(n)^4}{2\sqrt{\pi}t^{\frac{3}{2}}} \int_0^\infty \frac{dk_1}{2\pi} \frac{\Gamma^2(n+ik_1)\Gamma^2(n-ik_1) - \Gamma^4(n)}{k_1^4 \Gamma(n)^4} G(m,-k_1)G(m,k_1) =$$
$$= C_{III}(n,m)\langle g_n(t)\rangle. \quad \text{(A.44)}$$

To calculate the remaining terms from the 4th time ordering, we need to consider the following expression

$$f_{IV,U}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t}}{k_2^2 - k_1^2} G(n,k_1)G(n,-k_1)G(m,k_2)G(m,-k_2). \quad \text{(A.45)}$$

Let us evaluate $f_{IV,2} - f_{IV,U}$:

$$f_{IV,2}(t) - f_{IV,U}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} t}{k_2^2 - k_1^2} G(n,k_1)G(n,-k_1) \times$$
$$(G(m+ik_1,k_2)G(m-ik_1,-k_2) - G(m,k_2)G(m,-k_2)). \quad \text{(A.46)}$$

For $t \gg 1$ one has $k_1 \ll 1$ and

$$f_{IV,2}(t) - f_{IV,U}(t) =$$
$$-\frac{\Gamma^4(n)}{2\sqrt{\pi}t^{\frac{3}{2}}} \int_0^\infty \frac{dk_2}{2\pi} \frac{3\Gamma^2(-ik_2+m)\Gamma^2(ik_2+m)(\psi'(m-ik_2)+\psi'(m+ik_2))\sinh(2k_2\pi)}{\pi k_2} =$$
$$\equiv C_{IV,1}(n,m)\langle g_n(t)\rangle, \quad \text{(A.47)}$$

where ψ is digamma function. The next step is to evaluate

$$f_{IV,U}(t) - f_{Z,IV}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} t k_1^2}{(k_2^2 - k_1^2)k_2^2} G(n,k_1)G(n,-k_1)G(m,k_2)G(m,-k_2). \quad \text{(A.48)}$$

In this integral $k_1 \ll 1$ and

$$f_{IV,U}(t) - f_{Z,IV}(t) \approx \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} 4tk_1^4 \Gamma^4(n)}{(k_2^2 - k_1^2)k_2^2} G(m, k_2)G(m, -k_2). \quad (A.49)$$

To calculate the asymptotic, we add and subtract the following expression (below we will see that it is equal to zero):

$$\delta = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} 4tk_1^4 \Gamma^4(n)}{(k_2^2 - k_1^2)k_2^2} \Gamma^4(m) 4k_2^2. \quad (A.50)$$

This gives:

$$f_{IV,U}(t) - f_{Z,IV}(t) - \delta = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} 4tk_1^4 \Gamma^4(n)}{(k_2^2 - k_1^2)k_2^2} (G(m, k_2)G(m, -k_2) - \Gamma^4(m) 4k_2^2). \quad (A.51)$$

Here we can expand in k_1 and obtain

$$\begin{aligned} f_{IV,U}(t) - f_{Z,IV}(t) - \delta &= \tfrac{\Gamma^4(n)}{2\sqrt{\pi} t^{\frac{3}{2}}} \textstyle\int_0^\infty \frac{dk_2}{2\pi} \frac{3}{2k_2^4} (G(m, k_2)G(m, -k_2) - \Gamma^4(m) 4k_2^2) \\ &\equiv C_{IV,2}(n, m) \langle g_n(t) \rangle \end{aligned} \quad (A.52)$$

The last step to do is the calculation of δ:

$$\begin{aligned} \delta = 16t\Gamma^4(n)\Gamma^4(m) \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} k_1^4}{k_2^2 - k_1^2} &= \\ 8t\Gamma^4(n)\Gamma^4(m) \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} k_1^4 - e^{-k_2^2 t} k_2^4}{k_2^2 - k_1^2} &= \\ 8t\Gamma^4(n)\Gamma^4(m) \partial_t^2 \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2 t} - e^{-k_2^2 t}}{k_2^2 - k_1^2} &= \\ 8t\Gamma^4(n)\Gamma^4(m) \partial_t^2 \int_0^\infty \frac{dx}{2\pi} \int_0^\infty \frac{dy}{2\pi} \frac{e^{-x^2} - e^{-y^2}}{x^2 - y^2} &= 0. \end{aligned} \quad (A.53)$$

Finally we have:

$$\bar{f}_{IV}(t) \equiv f_{IV}(t) - f_Z(t) = (C_{III}(n, m) + C_{IV,1}(n, m) + C_{IV,2}(n, m)) \langle g_n(t) \rangle \quad (A.54)$$

where coefficients $C_i(n, m)$ are defined in Eqs.(A.41,A.47,A.52).

A.6 Contributions from regions II and V and the final result

The time regions II and V provides equal corrections to the Green function, so we will consider the region II only. Here the correction to the Green function is

$$f_{II}(t) = \int_0^t dt_3 \int_{-\infty}^0 \langle g_n(t,0) g_m(t_3,t_4)\rangle dt_3 dt_3 =$$
$$\int_0^\infty \frac{dk_1 dk_2 dk_3}{(2\pi)^3} \frac{e^{-k_2^2 t} - e^{-k_1^2 t}}{k_1^2 - k_2^2} \frac{G(n,k_1)G(m,-k_3)G(n+ik_2,k_3)G(m-ik_2,-k_1)}{\Gamma(2ik_2)\Gamma(-2ik_2)}$$
$$\times G^{2,4}_{4,4}\left(\left|\begin{smallmatrix} 1-ik_2-ik_1 & 1-ik_2+ik_1 & 1-ik_3 & 1+ik_3 \\ m & n-ik_2 & 1-m & 1-ik_2-n \end{smallmatrix}\right|, 1\right). \tag{A.55}$$

Here $G^{2,4}_{4,4}$ is Meijer G-function. In the limit $t \to \infty$ we can obtain the following asymptotic formula for this function:

$$f_{II}(t) = \langle g_n(t)\rangle (C_{II,1}(n,m) + C_{II,2}(n,m)) \tag{A.56}$$

where coefficients $C_i(n,m)$ are given by

$$\begin{aligned} C_{II,1}(n,m) &= \int_0^\infty \tfrac{dk_2 dk_3}{k_2 k_3 \pi^4 \Gamma(n)^2} \Gamma(-ik_2+m)^2 \Gamma(m-ik_3)\Gamma(m+ik_3)\Gamma(n+ik_2-ik_3)\Gamma(n+ik_2+ik_3) \\ &\times G^{2,4}_{4,4}\left(\left|\begin{smallmatrix} 1-ik_2 & 1-ik_2 & 1-ik_3 & 1+ik_3 \\ m & n-ik_2 & 1-m & 1-ik_2-n \end{smallmatrix}\right|, 1\right), \\ C_{II,2}(n,m) &= \int_0^\infty \tfrac{dk_1 dk_3}{k_1 k_3 \pi^4 \Gamma(n)^4} \Gamma(m-ik_1)\Gamma(m+ik_1)\Gamma(m+ik_3)\Gamma(m-ik_3)\Gamma(n-ik_1)\Gamma(n+ik_1) \\ &\times G^{2,4}_{4,4}\left(\left|\begin{smallmatrix} 1-ik_1 & 1+ik_1 & 1-ik_3 & 1+ik_3 \\ m & n & 1-m & 1-n \end{smallmatrix}\right|, 1\right). \end{aligned} \tag{A.57}$$

We combine now Eqs.(A.41,A.54,A.56,A.57) to obtain the complete result for the relative correction to the Green function:

$$\frac{\delta\langle G^{\frac{n}{\Delta}}\rangle}{\langle G^{\frac{n}{\Delta}}\rangle} = \frac{N\Gamma^2\Delta}{m} \frac{b^m}{\Gamma(2m)(2M)^{2m-2}} \times$$
$$[2C_{III}(n,m) + C_{IV,1}(n,m) + C_{IV,2}(n,m) + 2(C_{II,1}(n,m) + C_{II,2}(n,m))]. \tag{A.58}$$

Now we set $n = \frac{1}{4}$, $m = \frac{1}{2}$ and $\Delta = \frac{1}{4}$ in the above Eq.(A.58) and obtain the result for the first order correction to the Green function of the SYK_4 model in presence of SYK_2 perturbation, as it is presented in Eq.(11) of the main text.

Appendix B

Polaron Hamiltonian its eigenvalues and eigenfunctions.

B.1 The effective action

In this part, we will describe the solution of the problem using a geometrical approach. The logic will be the same as in the main text. We derive the effective action in adiabatic approximation and then the first non-adiabatic correction. Full action is provided in Eq.(B.22).

B.1.1 Adiabatic approximation

The action of the SYK model at the Hyperbolic plane (we use Poincaré disk model) was presented at the main text. After proper regularization it has the form:

$$S = \int_0^{\beta} \left\{ \frac{1}{2} g_{\mu\nu} \dot{X}^{\mu} \dot{X}^{\nu} - \gamma \omega_{\mu} \dot{X}^{\mu} \right\} d\tau - \frac{g\gamma}{4} \int_0^{\beta} d\tau_1 d\tau_2 \chi^{1/2}_{z(\tau_1), z(\tau_2)} \tag{B.1}$$

Here $g_{\mu\nu}$ is a metric tensor and ω_{μ} is the spin connection. And $\gamma = \alpha_S N$ We also introduced the following notations:

$$\beta = \frac{J\beta}{\gamma}, \qquad g = \frac{b^{2\Delta}}{2} \frac{N\Gamma^2}{J^2} \gamma^{2-4\Delta} = \frac{N\gamma}{4\sqrt{\pi}} \frac{\Gamma^2}{J^2}. \qquad \chi = \frac{(1-|z_1|^2)(1-|z_2|^2)}{|1-z_1^* z_2|^2} \tag{B.2}$$

Here z is a complex coordinate of the point at the model. We will use coordinates ξ and φ which are defined as $z = \tanh(\xi/2)e^{i\varphi}$ to solve our problem. We also perform Hubbard–Stratonovich transformation, as a result the action of the problem will be:

$$\begin{aligned} S_{SYK} &= \tfrac{1}{2}\int_0^\beta \left[\tfrac{\dot{\xi}^2}{2} + \sinh^2(\xi)\tfrac{\dot{\varphi}^2}{2} - \gamma\cosh(\xi)\dot{\varphi}\right] d\tau \\ S_\Phi &= \tfrac{1}{4g\gamma}\int d\mu\Phi(x)(-L - \tfrac{1}{4} + \delta^2)\Phi(x) \\ S_{int} &= \int_0^\beta \Phi(x(\tau))d\tau \end{aligned} \tag{B.3}$$

Here L is the Laplace operator and $d\mu$ is the invariant measure on the hyperbolic plane and we should take a limit $\delta \to 0$. If we integrate the bosonic field Φ we will obtain the previous action. We employ an adiabatic approximation, assuming that the motion along the phase φ is much slower than along radial coordinate ξ. Then functional integral over trajectories $\xi(\tau)$ can be done at fixed value of φ, which is the way to find an effective action for $\dot{\varphi}(\tau)$. Since parameter $\gamma \gg 1$, we can use saddle point approximation for $\dot{\varphi}$, which leads to the relation $\dot{\varphi} = \frac{\gamma\cosh(\xi)}{\sinh^2(\xi)}$. The effective action is then defined in the following way:

$$S_{eff}[\varphi(\tau)] = \ln\left(\int D\Phi D\xi\delta\left(\dot{\varphi} - \frac{\gamma\cosh(\xi)}{\sinh^2(\xi)}\right)e^{-S}\right) \tag{B.4}$$

A Lagrange variable $\lambda(\tau)$ is used to remove the δ-function. Then we need to calculate the functional integral with the action dependent of trajectories $\xi(\tau)$ and $\lambda(\tau)$:

$$S = S_\Phi + S_{int} + \int_0^\beta \left[\frac{\dot{\xi}^2}{2} - \frac{1}{2}\gamma^2\frac{\cosh^2(\xi)}{\sinh^2(\xi)} - \lambda(\tau)\left(\dot{\varphi} - \frac{\gamma\cosh(\xi)}{\sinh^2(\xi)}\right)\right] d\tau \tag{B.5}$$

$$\simeq S_\Phi + S_{int} + \int_0^\beta \left[\frac{1}{2}\dot{\xi}^2 - \lambda(\tau)\left(\dot{\varphi} - 2\gamma e^{-\xi(\tau)}\right)\right] d\tau - \int_0^\beta 2\gamma^2 e^{-2\xi(\tau)}d\tau \tag{B.6}$$

Representation (B.6) follows from Eq.(B.5) since the condition $\gamma \gg 1$ leads also to $\xi \gg 1$; we also omit irrelevant constant $\gamma^2/2$. Now calculation of the functional integral over $\xi(\tau)$ is reduced to the solution of the 1D quantum-mechanical problem with the Hamiltonian

$$H = -\frac{\partial_\xi^2}{2} + 2\gamma\lambda(\tau)e^{-\xi} + \Phi(\xi,\varphi(\tau)) \tag{B.7}$$

It is the same Hamiltonian as one presented in the main text. Its eigenfunctions and eigenvalues will be presented below. Last term in the action (B.6) was neglected in the Hamiltonian (B.7) due to its smallness w.r.t. other terms; however, we will need this term later. The term $\Phi(\xi,\varphi)$ in Eq.(B.7) came from S_{int} term in Eq.(B.6). Explicit form of $\Phi(\xi,\varphi)$ is to be obtained variationally. Variation of the full action over Φ leads to the relation

$$\Phi_0(\varphi,\xi) = -\int G_\Phi(\xi,\varphi|\xi',\varphi')\psi_g^2(\xi',\varphi')\frac{d\varphi'}{\varepsilon(\varphi')}d\xi' \tag{B.8}$$

where G_Φ is the Green function of the operator $-L - \frac{1}{4} + \delta^2$, and the limit $\delta \to 0$ is implied. Full analysis of this Green function is provided in Sec.IV below; here we need its asymptotic expression only (it coincides with Eq.(B.50) in the end of Sec.IV). $G_\Phi(\xi_1,\varphi_1|\xi_2,\varphi_2) = 2g\gamma\left(\frac{e^{-\xi_1-\xi_2}}{\varphi_{12}^2}\right)^{1/2}$, where $\varphi_{12} = 2\sin(\frac{\varphi_1-\varphi_2}{2})$.

Using Eq.(B.8) and the result of variation of the full action over $\lambda(\tau)$, we obtain, as explained in the main text:

$$\Phi_0(\xi,\varphi) = -\frac{\kappa\sqrt{\lambda\gamma}}{2}e^{-\xi/2} \quad \text{where } \lambda(\tau) = \frac{\kappa(\kappa-1)}{32\dot{\varphi}} \quad \text{and} \quad \kappa^2 = 32g\ln\left(\frac{\kappa\beta}{16\pi}\right) \tag{B.9}$$

We start our analysis of Eq.(B.7)) from the simplest case of $\dot{\varphi} = \varepsilon_0 \equiv 2\pi/\beta$. Then Schrodinger equation (B.7) with potential (B.9) allows for exact ground-state ψ_g and excited bound-state solutions ψ_n. We provide these functions below together with corresponding eigenvalues, assuming $\kappa > 1$:

$$\psi_g(\chi) = \frac{e^{-\chi/2}\chi^{\kappa/2-1/2}}{\sqrt{2\Gamma(\kappa-1)}}; \qquad E_g = -\frac{(\kappa-1)^2}{32} \tag{B.10}$$

$$\begin{aligned}\psi_n(\chi) &= \frac{1}{\sqrt{\frac{2\Gamma(n+1)\Gamma(\kappa-n)}{\kappa-2n-1}}}e^{-\chi/2}\chi^{(-1-2n+\kappa)/2}U(-n,-2n+\kappa,\chi);\\ E_n &= -\frac{(1+2n-\kappa)^2}{32}\end{aligned} \tag{B.11}$$

where $\chi = 8\sqrt{\gamma\lambda}e^{-\xi/2}$ and $U(n,m,\chi)$ is confluent hypergeometric function; line (B.11) is valid for $1+2n<\kappa$.

Now we need to generalize the above result for non-constant but slowly varying $\dot\varphi \equiv \varepsilon(\varphi)$. Our goal is to determine effective action $S_{eff}[\varphi(\tau)]$; equivalent representation can be obtained in terms of $S_{eff}[\varepsilon(\varphi)]$, since it is always assumed that $\dot\varphi \equiv \varepsilon(\varphi) > 0$. Formally, this functional can be written as

$$S_{eff}[\varphi(\tau)] = \left[S_\Phi + \int_0^\beta E_g(\lambda(\tau),\Phi)d\tau - \int_0^\beta \lambda(\tau)\dot\varphi d\tau\right]_{saddle} \tag{B.12}$$

where "saddle" means that Φ and λ should be determined from the saddle point equations.

To find the energy of the ground state for a general choice of $\varepsilon(\varphi)$ it is convenient to consider three terms in the Hamiltonian (B.7) separately and notice that the term which contains $\lambda(\varphi)$ is canceled out in the effective action (B.12). Then we need to calculate the average of the two other terms in the Hamiltonian over the deformed (dependent on $\varepsilon(\varphi)$) ground state:

$$\tilde{E}_g = \frac{\kappa-1}{32} - \int G_\Phi(\xi,\varphi|\xi',\varphi')\psi_g^2(\xi',\varphi')\psi_g^2(\xi,\varphi)\frac{d\varphi'}{\varepsilon(\varphi')}d\xi' d\xi \tag{B.13}$$

The first term in (B.13) comes from kinetic term in the Hamiltonian (B.7), its dependence on $\varepsilon(\varphi)$ is weak and we neglect it in the following. We will estimate its influence below. The second term, together with S_Φ term in Eq.(B.12), combine to our final result for the action in the adiabatic approximation:

$$\begin{aligned} S_{eff} = -\frac{1}{2}\int G_\Phi(\xi,\varphi|\xi',\varphi')\psi_g^2(\xi',\varphi')\psi_g^2(\xi,\varphi)\frac{d\varphi' d\varphi}{\varepsilon(\varphi')\varepsilon(\varphi)}d\xi' d\xi = \\ -\frac{g}{2}\int \frac{\kappa-1}{\kappa}\left(\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^2}\right)^{1/2}\frac{d\varphi_1 d\varphi_2}{\varepsilon(\varphi_1)\varepsilon(\varphi_2)} \end{aligned} \tag{B.14}$$

For the applicability of our adiabatic approximation strong inequality $\kappa \gg 1$ is needed, thus $\frac{\kappa-1}{\kappa} \approx 1$.

B.1.2 Main non-adiabatic correction

The aim of this Section is to find the first non-adiabatic correction to the action. This correction is due to virtual transitions between the levels of the 1D quantum mechanical problem with the Hamiltonian (B.7) which describes motion along coordinate ξ. General form of such a correction to S_{eff} is

$$\delta S_{eff} = \left[\sum_n \int_0^\beta d\tau \frac{(\partial_\tau H)_{ng}(\partial_\tau H)_{gn}}{(E_n(\tau) - E_g(\tau))^3} \right]_{saddle} \tag{B.15}$$

Here E_n is an energy of the excited state n which adiabatically depends on τ and $(\partial_\tau H)_{ng}$ is a matrix element of the operator $\partial_\tau H$ between ground state and n-th state. Equation (B.15) can be obtained applying quantum-mechanical perturbation theory with respect to time-dependent terms in the Hamiltonian. The expression (B.15) comes in the next order after the Berry phase term.

To employ general form (B.15) for our purpose, it is convenient to introduce the following notations:

$$M_{n\alpha} = \int_0^\infty \psi_n(\chi)\psi_g(\chi)\chi^\alpha \frac{2d\chi}{\chi} = \frac{1}{\sqrt{\frac{\Gamma(n+1)\Gamma(\kappa-n)\Gamma(\kappa-1)}{\kappa-2n-1}}} \frac{\Gamma(-1-n+\kappa+\alpha)\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{B.16}$$

In the limit $\kappa \gg 1$ we have: $M_{n\alpha} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}\kappa^{\alpha-n/2}$. Time derivative $\partial H/\partial\tau$ can be written in the form

$$\partial_\tau H = 2\gamma\partial_\tau\lambda e^{-\xi} - \frac{\kappa\sqrt{\gamma\lambda}\partial_\tau\lambda}{4\lambda} e^{-\xi/2} = \frac{\partial_\tau\lambda}{32\lambda}\left(\chi^2 - \kappa\chi\right) \tag{B.17}$$

Using Eq.(B.17) and notations (B.16) we write:

$$(\partial_\tau H)_{gn} = \frac{1}{32}\frac{\partial_\tau\lambda}{\lambda}(M_{n2} - \kappa M_{n1}) = \frac{1}{32}\frac{\partial_\tau\lambda}{\lambda} n\kappa^{2-n/2}\sqrt{\Gamma(n+1)} \tag{B.18}$$

Here the limit of large κ was used to obtain the last result. As $E_n = -\frac{1}{32}(-\kappa+2n+1)^2$ and $\kappa \gg 1$ the leading contribution to the S_{eff} comes from the first term in the sum. It brings us to the following expression:

$$\delta S_{eff} = \frac{1}{2}\int_0^{\beta}\left(\frac{\partial_\tau \lambda}{\lambda}\right)^2 d\tau = \frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\varepsilon(\varphi)}\left(\partial_\varphi \varepsilon(\varphi)\right)^2 \tag{B.19}$$

The last expression follows from the expression for λ in (B.9).

Now we recall the last term in the action (B.6), which was not taken into account in the adiabatic approximation. In the limit of large κ the contribution of this term into the ground-state energy can be evaluated as $-2\gamma^2\int d\xi \psi_g^2(\xi)e^{-2\xi}$. Thus its contribution to the effective action is

$$\delta S = -\frac{1}{2}\int_0^{\beta}\int d\xi \psi_g^2(\xi)(2\gamma e^{-\xi})^2 \approx -\frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\varepsilon(\varphi)}\varepsilon^2(\varphi) \tag{B.20}$$

Combining the terms in Eqs.(B.19,B.20) we find total non-adiabatic contribution to the action

$$\delta S_{eff} = -\int_0^{\beta} Sch\left\{e^{i\varphi(\tau)}, \tau\right\} d\tau \tag{B.21}$$

which exactly reproduces the Schwarzian action known for the SYK_4 theory. Full action is given by the sum of Eq.(B.21) and Eq.(B.14):

$$S_{eff} = \frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\varepsilon(\varphi)}\left(\left(\partial_\varphi\varepsilon(\varphi)\right)^2 - \varepsilon(\varphi)^2\right) - \frac{g}{2}\int\left(\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^2}\right)^{1/2}\frac{d\varphi_1 d\varphi_2}{\varepsilon(\varphi_1)\varepsilon(\varphi_2)} \tag{B.22}$$

In the next Section we will evaluate fluctuations of $\varepsilon(\varphi)$ controlled by the action (B.22).

B.2 Fluctuation corrections

In the Section we analyze Gaussian fluctuations of the function $\varepsilon(\varphi)$ using the action provided in Eq.(B.22), and estimate corrections to the fermion Green function related to these fluctuations.

B.2.1 Gaussian fluctuations of the $\varepsilon(\varphi)$ function

Consider the 2nd-order expansion of the action over Fourier-components $\delta\varepsilon_m$ defined as

$$\varepsilon(\theta) = \varepsilon_0 + \frac{1}{2\pi}\sum_m \delta\varepsilon_m e^{im\theta} \tag{B.23}$$

We will assume $\delta\varepsilon(\theta) \ll \varepsilon_0$; equivalently, we write $\varphi = \theta + u(\theta)$ and $u(\theta) \ll 1$. Do derive the action up to quadratic terms in fluctuations, we need to expand $\varepsilon(\varphi)$ up to a second order:

$$\varepsilon(\varphi) = \varepsilon_0 \frac{d\varphi}{d\theta} = \varepsilon_0(1 + u'(\theta)) \approx \varepsilon_0(1 + u'(\varphi) - u(\varphi)u''(\varphi)) \tag{B.24}$$

The first term in Eq.(B.22) leads to:

$$\begin{aligned} \frac{1}{2}\int_0^{2\pi} \frac{d\varphi}{\varepsilon(\varphi)} \left((\partial_\varphi \varepsilon(\varphi))^2 - \varepsilon(\varphi)^2\right) &\approx \frac{\varepsilon_0}{2}\int_0^{2\pi} d\varphi \left((u'')^2 - (1 + u'u')\right) \\ &= \frac{1}{4\pi\varepsilon_0}\sum_m \delta\varepsilon_m \delta\varepsilon_{-m}(m^2 - 1) \end{aligned} \tag{B.25}$$

The second term in Eq.(B.22) is not quite trivial to handle, since the integral over $(\varphi_1 - \varphi_2)$ formally diverges, so some regularization is needed. Explicit regularization with invariant short-scale cut-off $\varphi_{12}^2/\varepsilon(\varphi_1)\varepsilon(\varphi_2) > l$ can be used to demonstrate that higher harmonics ε_m are free from this log-divergence. Since this calculation is relatively cumbersome, we present here simpler derivation based on dimensional regularization. Namely, we replace power $\frac{1}{2}$ in the 2-nd term in (B.22) by some $d < \frac{1}{2}$ and then take the limit $d \to \frac{1}{2} - 0$. At $d < \frac{1}{2}$ straightforward Fourier-transformation leads to (with the accuracy up to terms quadratic in ε_m):

$$\begin{gathered} \tfrac{g}{4\gamma}\int \left(\tfrac{\varepsilon(\varphi)\varepsilon(\varphi')}{\varphi_{12}}\right)^d \tfrac{d\varphi' d\varphi}{\varepsilon(\varphi')\varepsilon(\varphi)} = \\ \tfrac{1}{2}\tfrac{g}{4\gamma}\sum_{m\neq 0} \tfrac{u_m u_{-m}}{m^2}\int_0^{2\pi} \tfrac{d\varphi}{2\pi} 2(d-1)\varepsilon_0^{2d-4}\left(\tfrac{1}{4\sin^2(\varphi)}\right)^d ((d-1)\cos(2m\varphi) + d) \end{gathered} \tag{B.26}$$

Then last integral in Eq.(B.26) can be calculated using the following formula:

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \left(\frac{1}{4\sin^2(\varphi)} \right)^d e^{2im\varphi} = \frac{1}{2\cos(\pi d)} \frac{\Gamma(m+d)}{\Gamma(2d)\Gamma(1+m-d)} \tag{B.27}$$

where m is any integer number. We are interested in the m-dependent coefficients which are obtained by derivative of the ratio $\Gamma(m+d)/\Gamma(m+1-d)$ over d, evaluated in the limit $d \to \frac{1}{2}$. The result reads

$$S_{eff} \approx \frac{1}{4\pi\varepsilon_0} \sum_m \delta\varepsilon_m \delta\varepsilon_{-m}(m^2-1) + \frac{g}{2} \sum_m \frac{\tilde{\psi}(m)}{4\pi\varepsilon_0^3} \delta\varepsilon_m \delta\varepsilon_{-m} \tag{B.28}$$

Here $\tilde{\psi}(x) = \Psi(x+1/2) - \Psi(-1/2)$ and $\Psi(x) = (\ln\Gamma(x))'$ is the digamma function. This action leads to the following correlation function:

$$\langle \delta\varepsilon_m \delta\varepsilon_{-m} \rangle = \frac{2\pi\varepsilon_0^3}{\varepsilon_0^2(m^2-1) + \frac{g}{2}\tilde{\psi}(m)} \tag{B.29}$$

We use it below for calculations of the corrections to fermion Green function.

B.2.2 Estimation of the fluctuations of the kinetic term

The contribution to the action from the kinetic term has the form:

$$S_{kin} = \int \frac{\kappa}{32} d\tau \qquad \kappa^2 = 32g \ln\left(\frac{\kappa}{8\varepsilon(\varphi)} \right) \tag{B.30}$$

Assuming smallness of fluctuations we can write $\kappa = \kappa_0 + \delta\kappa$ where κ_0 is defined by $\varepsilon(\varphi) = \varepsilon_0$. We will also define a parameter $\alpha = \frac{32g}{\kappa_0^2} \ll 1$. The connection between $\delta\kappa$ and $\delta\varepsilon$ can be obtained from the definition of κ and has the form:

$$\delta\kappa = \frac{\kappa_0}{2} \left(\frac{\alpha}{2} \left(\frac{\delta\varepsilon}{\varepsilon_0} \right)^2 - \alpha \frac{\delta\varepsilon}{\varepsilon_0} \right) \tag{B.31}$$

This expression leads to the following form of the above action:

$$S_{kin} = \frac{1}{2\pi} \frac{g}{2\kappa_0} \sum_n \frac{\delta\varepsilon_n \delta\varepsilon_{-n}}{\varepsilon_0^2} \tag{B.32}$$

One can see smallness of this part due to the factor $\frac{1}{\kappa\varepsilon_0} \ll 1$ with respect to the second term in the ($B.28$)

B.2.3 Correction to the Green function

Fermion Green function can be obtained as an average of the field $\hat{G}(\theta_1, \theta_2)$, evaluated with the effective action (B.22), where

$$\hat{G}(\theta_1, \theta_2) = -\left(b\gamma^2 \frac{\varepsilon(\theta_1)\varepsilon(\theta_2)}{4\sin^2\left(\frac{\varphi(\theta_1)-\varphi(\theta_2)}{2}\right)} \right)^{\Delta} \tag{B.33}$$

The saddle point approximation ($\varphi(\theta) = \theta$) leads to $\langle \hat{G}(\theta_1, \theta_2)\rangle = G_c = -\left(b\gamma^2 \frac{\varepsilon_0^2}{\theta_{12}^2}\right)^{\Delta}$. We are interested in the quadratic correction to the Green's function. So we need to find the second-order correction by $\delta\varepsilon$ to $\hat{G}$:

$$\begin{gathered}
\frac{\delta\hat{G}(\theta_1,\theta_2)}{G_c(\theta_1,\theta_2)} = \frac{1}{2}\sum_{m\neq\pm1,0} \langle\delta\varepsilon_m \delta\varepsilon_{-m}\rangle O_m(\theta_1 - \theta_2) \\
O_m(\theta) = -\frac{\Delta}{(2\pi)^2 \sin^2\left(\frac{\theta}{2}\right)\varepsilon_0^2 m^2} \left((\Delta(1-m^2)+1)\cos(m\theta) \right. \\
+\cos(\theta)\left((\Delta-1)m^2 - \Delta + \Delta\left(m^2+1\right)\cos(m\theta)\right) \\
\left. -\Delta\left(m^2+1\right) + m^2 + 2\Delta m \sin(\theta)\sin(m\theta) - 1 \right)
\end{gathered}$$

For large κ only terms with large m will be important. In this case: $O_m(\theta) = \frac{2\Delta}{(2\pi\varepsilon_0)^2}(\Delta - 1 + \Delta\cos(m\theta)) \sim \frac{2\Delta}{(2\pi\varepsilon_0)^2}$ so we can write

$$\frac{\delta\hat{G}(\theta_1, \theta_2)}{G_c(\theta_1, \theta_2)} \sim \frac{1}{2}\frac{2\Delta}{(2\pi\varepsilon_0)^2} \sum_{m\neq\pm1,0} \langle\delta\varepsilon_m \delta\varepsilon_{-m}\rangle = \frac{1}{2}\frac{2\Delta}{2\pi} \sum_{m\neq\pm1,0} \frac{\varepsilon_0}{\varepsilon_0^2(m^2-1) + \frac{g}{2}\tilde{\psi}(m)} \sim \frac{\Delta}{\pi}\frac{1}{\varepsilon_0 m_*} \tag{B.34}$$

Here m_* is defined us $\varepsilon_0^2(m_*^2 - 1) = \frac{g}{2}\tilde{\psi}(m_*)$. For large κ we can write, using Eq.(B.9): $\varepsilon_0 m_* = \frac{\kappa}{8}$, thus corrections to fermion Green function are small at any θ.

B.3 Higher orders of the fermionic Green function.

The major object of our theory is the Majorana Green function $G(\tau)$ averaged over disorder variables which enter the Hamiltonian, Eq.(1) of the main text. However, local Majorana Green function $G_i(\tau,\tau') = -\langle \chi_i(\tau)\chi_i(\tau')\rangle$ contains more information about system's dynamics.

One of the methods to extract this additional information is to consider higher-order Green functions, defined below:

$$G^{(p)}(\tau,\tau') \equiv \langle \left(-\frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau') \right)^p \rangle \tag{B.35}$$

Here we restrict ourselves by the region of moderately high $p \ll N$, where it is easy to show that

$$G^{(p)}(\tau_1,\tau_2) = (-1)^p \left\langle \left[b\frac{e^{-\xi_1-\xi_2}}{\sin^2(\frac{1}{2}(\varphi_1-\varphi_2))} \right]^{\Delta p} \right\rangle = (-1)^p C^2_{\Delta p} \left\langle \left[\frac{b}{4\gamma}\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\sin^2(\frac{1}{2}(\varphi_1-\varphi_2))} \right]^{\Delta p} \right\rangle_{S_\varphi} \tag{B.36}$$

Angular brackets in the middle formula of the above equation mean averaging over quantum action S_{eff}, see Eq.(11) of the main text. Formula in the R.H.S. of (B.36) is obtained after we take average over fluctuations of ξ_1 and ξ_2 over the polaron ground state $\psi_g(\xi)$, where C_α is defined below:

$$C_\alpha = \left(\frac{2\gamma}{\varepsilon(\varphi)}\right)^\alpha \int e^{-\alpha\xi}\psi_g^2(\xi,\varphi)d\xi = \frac{\Gamma(\kappa+2\alpha-1)}{\Gamma(\kappa-1)}\kappa^{-\alpha}(\kappa-1)^{-\alpha} \approx \exp\left(\frac{2\alpha^2}{\kappa}\right) \tag{B.37}$$

We used assumption $\alpha \ll \kappa$ to make the last approximation. Final averaging over S_φ in the R.H.S. of Eq.(B.36) should be done with the full phase-dependent action given by Eq.(B.22). Last expression in Eq.(B.37) is valid in the main order of approximation for $\kappa \gg 1$ and $\alpha \gg 1$.

Consider now the effect of integration over fluctuations of angular modes $\varepsilon(\varphi)$

and define relevant measure for these fluctuations

$$g_p(\tau_1,\tau_2)=\frac{\langle G^{(p)}(\tau_1,\tau_2)\rangle}{C^2_{\Delta p}G^p_c(\tau_1,\tau_2)}=\langle\exp\left[\Delta p\delta g(\theta_1,\theta_2)\right]\rangle=\exp\left(\frac{(\Delta p)^2}{2}\langle(\delta g(\theta_1,\theta_2))^2\rangle\right) \tag{B.38}$$

where $G_c(\tau_1,\tau_2)$ is the conformal saddle-point Green function, while the function $\delta g(\theta_1,\theta_2)$ is defined via the relation

$$\begin{aligned}\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{4\sin^2(\frac{\varphi_1-\varphi_2}{2})}\cdot\left[\frac{\varepsilon_0\varepsilon_0}{4\sin^2(\frac{\theta_1-\theta_2}{2})}\right]^{-1}&\equiv 1+\delta g(\theta_1,\theta_2)=\\ 1+u'(\theta_1)+u'(\theta_2)+\cot\left(\frac{\theta_1-\theta_2}{2}\right)&(u(\theta_2)-u(\theta_1))\end{aligned} \tag{B.39}$$

We use here definitions $\varphi=\theta+u(\theta)$ and $\varepsilon(\varphi)=\varepsilon_0\frac{d\varphi}{d\theta}$. To calculate the average in the R.H.S. of Eq.(B.38) we need to expand the R.H.S. of Eq.(B.39) up to linear terms in $u(\theta)$ and then use Fourier series:

$$\delta g(\theta_1,\theta_2)=\frac{1}{2\pi}\sum_m\left(ime^{im\theta_1}+ime^{im\theta_1}+\cot\left(\frac{\theta_1-\theta_2}{2}\right)\left(e^{im\theta_2}-e^{im\theta_1}\right)\right)u_m \tag{B.40}$$

Now we can average R.H.S. of Eq.(B.38) in the Gaussian approximation, using representation (B.40) and correlation function defined in (B.29). Correlation function in the θ-representation is (below $\theta=\theta_1-\theta_2$):

$$\langle\delta g^2(\theta_1,\theta_2)\rangle=\tfrac{1}{(2\pi)^2}\sum_m\left(2m\cos\left(\tfrac{m\theta}{2}\right)-2\cot\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{m\theta}{2}\right)\right)^2\langle u_m u_{-m}\rangle \tag{B.41}$$

$$\begin{aligned}&\approx\tfrac{1}{2\pi\varepsilon_0}\Re\sum_{m\neq0,\pm1}\tfrac{1}{m^2}\tfrac{1}{m^2+m_*^2}\left[2m^2\left(1+e^{im\theta}\right)+4im\cot\left(\tfrac{\theta}{2}\right)e^{im\theta}\right.\\ &\qquad\left.+2\cot^2\left(\tfrac{\theta}{2}\right)\left(1-e^{im\theta}\right)\right]\end{aligned} \tag{B.42}$$

$$=\tfrac{1}{\varepsilon_0}\tfrac{(2+m_*\theta)}{m_*^3\theta^2}\left[2m_*\theta\cosh\left(\tfrac{m_*\theta}{2}\right)-4\sinh\left(\tfrac{m_*\theta}{2}\right)\right]\exp\left\{-\tfrac{m_*\theta}{2}\right\}\equiv\tfrac{8}{\kappa}f(\theta)$$

where $\varepsilon_0 m_*=\kappa/8$ and last equality just defines a convenient notation. Asymptotic limits for the function $f(\theta)$ are given by

$$f(\theta) = \begin{cases} 1 & m_* \theta \gg 1 \\ \frac{\theta m_*}{3} & m_* \theta \ll 1 \end{cases} \tag{B.43}$$

Finally, combining Eqs.(B.36,B.37,B.38,B.41) and replacing $\Delta \to \frac{1}{4}$ we obtain

$$\frac{G^{(p)}(\tau_1, \tau_2)}{[G(\tau_1, \tau_2)]^p} = \exp\left[\frac{p^2}{4\kappa}\left(1 + f(\theta_{12})\right)\right] \tag{B.44}$$

B.4 The Green function of the boson field on the hyperbolic plane.

The action of the bosonic field is

$$S_\Phi = \frac{1}{2g}\int d\mu \Phi(x)(-L - \frac{1}{4} + \delta^2)\Phi(x) \tag{B.45}$$

Here L is the Laplace operator and $d\mu$ is an invariant measure on the hyperbolic plane and $\delta \to 0$. We use the Poincaré disk model. The Green function of the bosonic field satisfy the following equation:

$$(-L - \frac{1}{4} + \delta^2)G(z_1, z_0) = g\frac{\delta(z_1 - z_0)}{\sqrt{g(x_0)}} \tag{B.46}$$

All objects here are invariant under $SL(2,R)$ transformations so let us use transforms which maps $z_0 \mapsto 0$ in this case $z_1 \mapsto \frac{z_1 - z_0}{1 - z_1 \bar{z}_0}$. In new coordinates the form of equation will be the same but δ function will be localized in the origin of the hyperbolic plane so we expect the rotation invariant solution. It leads us to the equation:

$$\left[-(1-u)^2(u\partial_u^2 + \partial_u) - \frac{1}{4} + \delta^2\right]G(z) = g\frac{\delta(u)}{4\pi} \tag{B.47}$$

Here $u = |z|^2$. This equation can be written as the homogeneous equation with boundary conditions: the Green function should decay faster than $(1-u)^{1/2}$ at $u \to 1$, while at $u \ll 1$ it should behave as $G(u) \to -\frac{\ln(u)}{4\pi}$. Then we come to the following result:

$$G(u) = g\frac{1}{4}(1-u)^{\frac{1}{2}+\delta}{}_2F_1\left(\frac{1}{2}+\delta, \frac{1}{2}+\delta, 1+2\delta, 1-u\right) \tag{B.48}$$

Here ${}_2F_1(a, b, c; x)$ is a hypergeometric function. In the limit $\delta \to 0$

$$G(z_1, z_0) = g\frac{\sqrt{w}K(w)}{2\pi} \quad \text{where} \quad w = \frac{(1-|z_1|^2)(1-|z_0|^2)}{(1-z_1\bar{z}_0)(1-z_0\bar{z}_1)} \tag{B.49}$$

Here $K(w)$ is the complete elliptic integral of the first kind. In the limit $w \to 0$ we have:

$$G_\Phi(z_1, z_0) \approx \frac{g}{4}w^{1/2} \tag{B.50}$$

It is the last form (B.50) for the Bose field Green function G_Φ, which we use in the main text and in Sec.I above.