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QUANTUM KDV CHARGES, 2D CONFORMAL THEORIES AND EIGENSTATE THERMALIZATION HYPOTHESIS

Doctoral Thesis

by

KIRILL PAVLENKO

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Supervisor
Associate Professor Anatoly Dymarsky

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I hereby declare that the work presented in this thesis was carried out by myself at Skolkovo Institute of Science and Technology, Moscow, except where due acknowledgement is made, and has not been submitted for any other degree.

Candidate (Kirill Pavlenko)

Supervisor (Ass. Prof. Anatoly Dymarsky)

Abstract

PhD thesis is devoted to analytical study of Eigenstate Thermalization Hypothesis (ETH) in 2d CFTs. In this thesis we find spectrum of quantum KdV charges in large central charge limit by combining classical expressions for KdV charges in terms of action variables and quantum "corrections" which we unambiguously fix from the analytical form of quantum KdV charges acting on highest weight states. With this result we compute Generalized Gibbs Ensemble of 2D CFTs in large central charge limit. These calculations allows us to explicitly probe ETH in 2d CFTs and establish the correct form of thermalization hypothesis for large central charge theories, which we formulate in terms of Generalized ETH (GETH). We analytically check this hypothesis for several descendants in the "identity block" of 2D CFTs.

Publications

1. A. Dymarsky and K. Pavlenko, ”Generalized Gibbs Ensemble of 2d CFTs at large central charge in the thermodynamic limit”, J. High Energ. Phys. (2019) 2019: 98

2. A. Dymarsky and K. Pavlenko, ”Exact generalized partition function of 2D CFTs at large central charge”, J. High Energ. Phys. (2019) 2019: 77

3. A. Dymarsky and K. Pavlenko, ”Generalized Eigenstate Thermalization Hypothesis in 2D Conformal Field Theories”, Physical Review Letters 123 (11), 111602 (2019)

4. A. Dymarsky, K. Pavlenko and D. Solovyev, ”Zero modes of local operators in 2d CFT on a cylinder”, J. High Energ. Phys. (2020) 2020: 7

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Contents

1 Introduction

Emergence of statistical thermodynamics from quantum mechanics is one of the exciting open questions of theoretical physics. Extensive research in this field was carried out in recent years. It has became an accepted paradigm that universality of thermal equilibrium can be explained in terms of Eigenstate Thermalization Hypothesis (ETH) ıın case of nonintegrable (or "chaotic") systems, i.e., those without an extensive number of local conserved quantities. The ETH is an expectation that local properties of individual microstates – energy eigenstates of spatially extended chaotic quantum system – may only depend on thermodynamically relevant quantities, i.e. in most cases only on energy density of the microstate.

The idea eigenstate thermalization for quantum many-body systems was formulated in the works of Deutsch [1] and Srednicki [2]. The modern formulation [3] of the Eigenstate Thermalization Hypothesis (ETH) ansatz states that for any finite-support observable A

$$A_{ij} = \delta_{ij} f_A(E_i) + e^{-S/2} f(E_i, E_j) r_{ij}, \qquad A_{ij} \equiv \langle E_i | A | E_j \rangle, \tag{1.1}$$

where f_A is a smooth function which depends on energy density $f_A(E_i) = \tilde{f}_A(E_i/V)$, and function $\tilde{f}_A$ approaches some volume-independent shape in the thermodynamic limit, $V \to \infty$, $E/V =$ fixed. Here smoothness of f_A is simply the reflection of physical equivalence of different eigenstates with the same (exponentially close) values of energy density. As a trivial consequence of (1.1), provided most eigenstates obey the ETH ansatz, $f_A(E)$ will be equal to the microcanonical average, up to volume-suppressed corrections. Similarly, second term in (1.1), which describes off-diagonal matrix elements manifests indistinguishably of different eigenstates $|E_i\rangle$ with the same energy density in statistical sense by declaring average magnitude of $|A_{ij}^2|$ to be smoothly dependent only on mean energies E_i, E_j. The exponential suppression factor in (1.1) is purely kinematic and can be recovered in full generality independently of the details of the observable A [3].

Provided energy spectrum is non-degenerate, time-averaged expectation value of an observable A is given by the so-called diagonal ensemble, which only includes diagonal matrix elements A_{ii}. Agreement between the latter and the equilibrium expectation values emerging in model systems after a quantum quench [4] motivated substantial interest in Eigenstate Thermalization in the context of cold atomic systems.

When the system, besides energy, possesses a number of additional local (or quasi-local [5]) conserved quantities, as is normally the case for integrable systems, densities of these additional conserved charges are also thermodynamically relevant. In this case

standard formulation ETH generally does not apply. Accordingly emerging equilibrium can be different from the Gibbs state. The main hypothesis is that in this case the equilibrium can be described by the generalized Gibbs ensemble (GGE) [6–8], a generalization of the grand canonical ensemble that includes all the conserved charges. Validity of the GGE has been related to generalized eigenstate thermalization, which generalizes ETH to include all the conserved extensive quantities.

Most of the research in both integrable and nonintegrable cases were carried out via numerical methods. The aim of this thesis is to probe ETH analytically. In order to do so we focus on two-dimensional conformal field theories, where many analytical tools were developed before.

Conformal Field Theories in two dimensions describe one dimensional systems at criticality, most notably string theoretic worldsheet. Thermalization of two-dimensional theories came to the focus of attention recently in the context of entanglement spreading following a quantum quench. Additional motivation to study 2d theories comes from holographic duality, which relates thermal properties of boundary CFTs to black hole physics in AdS_3.

Due to tremendously difficulty of analytical checks, detailed understanding of ETH is still missing. This thesis is devoted to analytical study of ETH for 2d conformal field theories. Our project is the first non-trivial analytical study of ETH for space-extended quantum systems. In order to tackle ETH we had to study integrable structure of 2d CFTs, namely, quantum KdV charges. Large part of the thesis is devoted to a careful analysis of quantum integrability of 2d CFTs. We first establish the spectral properties quantum KdV charges which allows us study Generalized Gibbs ensemble of 2d CFTs. These results lead to an explicit analytic checks of generalized ETH, which are the main contribution of the thesis.

Main results of the thesis are the following:

- We explicitly find quantum spectrum of KdV charges up to first three orders in $1/c$ central charge expansion.
- We calculate Generalized partition function of 2d CFTs decorated by all quantum KdV charges up to first two orders in $1/c$ central charge expansion.
- We formulate the correct form of eigenstate thermalization for 2d CFTs, which must be in the form of generalized ETH which equates (in physical sense) individual states not to Gibbs ensemble but to Generalized Gibbs Ensemble.
- We resolve the issue found in previous works of discrepancy between thermal properties of heavy primary states and ETH prediction for the operators of so-called identity block. We show that heavy primary states are not equivalent to

Gibbs ensemble but to a singular Generalized Gibbs Ensemble. This conclusion is based on a detailed calculation involving explicit form of GGE partition function.

The thesis is organized as follows.

In section 2 we give a concise introduction to classical KdV hierarchy and Novikov's finite-zone solutions. In section 2.1 we consider a Schrodinger eigenvalue problem and study its periodic real-valued potentials. We show that isospectral deformations of the problem are generated via the Hamiltonian flows associated with the quantities Q_{2k-1} called classical KdV charges. In section 2.2 we define finite-zone potentials as solutions of the static generalized KdV equations. We show how the given spectral data defines a quasi-momentum on a Riemann curve. In section 2.3 we give an exact solution of generalized KdV equations for one-zone potential. In that case the Riemann curve is a torus and the solution can be found in terms of Weierstrass's function associated with the torus. In section 2.4 we take a different route and introduce perturbative approach of zone "openings". In sections 2.5 we find perturbatively the expression for KdV charges Q_{2k-1} for one-zone potential in terms of co-adjoint orbit invariant h, k and torus variable τ. We do the same for action variable I_{2k-1}. Then we re-express Q_{2k-1} in terms of I_{2k-1} and h. In section 2.6 we extend this result to two-zone petentials.

In section 3 we employ quasi-classical quantization to calculate spectrum of quantum KdV charges in the limit of large central charge c. Quantum-mechanically KdV charges expressed in terms of action variables become the expansion in $1/c$, while action variables become boson occupation numbers n_i. Crucially, classical expression, which is homogeneous in I_i, acquire quantum corrections which include terms of subleading powers. In section 3.1 we give a flash introduction to co-adjoint orbits of Virasoro algebra. We consider the group of diffeomorphisms of S^1 on periodic potential and wave-function satisfying Schrodinger equation. We show that action of diffS^1 foliates the space of all potentials into co-adjoint orbits of Virasoro algebra. In section 3.2 we introduce quantum KdV charges as symmetries of 2d Conformal field theories. In section 3.3 we discuss the space of states of KdV charges and introduce KdV-basis in "boson representation". In section 3.4 we find perturbatively the values of quantum KdV charges acting on primary states. We do this via Ordinary Differential Equations/ Integrable Models correspondence (ODE/IM) which relates KdV spectrum to solutions of an auxiliary Schrodinger equation. In section 3.5 we conjecture that quantum spectrum can be unambiguously fixed from the analytic form of Q_{2k-1} acting on the highest weight (primary) states and classical expressions found in sections 2.5 and 2.6. We confirm this and find the explicit expressions for the spectrum of Q_{2k-1} up to first three orders in $1/c$ expansion.

In section 4 we compute Generalized Partition function of 2d CFT decorated with

all qKdV charges as a function of infinite number of chemical potentials in the limit of large central charge. Section 4.1 is devoted to the standard partition function of 2d CFT in thermodynamic limit. We calculate the partition function via saddle point approximation, which is exact in thermodynamic limit. In section 4.2 we add higher KdV charges to the partition function and take large central charge limit $c \to \infty$. We find the partition function in that limit in a concise form. In section 4.3 we consider $1/c$ corrections from Q_3 and Q_5. Given the result for quantum spectrum obtained in section 2.5 we compute the partition function at first non-trivial $1/c$ correction with Q_3 and Q_5 turned on. We also find effective action at $1/c$. In section 4.4 we combine all the above results and calculate first two $1/c$ contributions to Generalized partition function with all quantum KdV charges included.

In section 5 we pedagogically develop the machinery of calculating zero modes of local operators from the vacuum family and calculate explicit expressions for all quasi-primaries of the dimension less or equal than eight. We also calculate the explicit expression for Q_7. Section 5.1 is devoted to a general technique of finding zero modes of the product of arbitrary local operators in terms of Stirling numbers of second kind. In section 5.2 we pedagogically show how to apply this technique to calculation of Q_3. In section 5.3 we continue to work out zero modes of various quasi-primaries, descending from the identity operator. In section 5.4 we calculate an explicit expression for Q_7 in terms of Virasoro operators.

In section 6 we introduce Eigenstate Thermalization Hypothesis (ETH) and state the hypothesis for 2d CFTs. Next, we compare generalized ensemble with the "eigenstate ensemble" that consists of a single primary state. At infinite central charge the ensembles match at the level of expectation values of local operators for any values of qKdV fugacities. When the central charge is large but finite, for any values of the fugacities the aforementioned ensembles are distinguishable. In section 6.1 we state the standard formulation of Eigenstate Thermalization Hypothesis. Section 6.2 is devoted to the challenge of formulating ETH for 2d CFTs. We argue that Generalized Gibbs Ensemble plays a crucial role for ETH in 2d CFTs. In section 6.3 we show that at infinite central charge "eigenstate ensemble" matches generalized Gibbs ensemble. We continue the investigation in section 6.4 where we show that this is no longer true at finite central charge. We give an argument that shows that elimination of discrepancy between GGE and "eigenstate ensemble" is not possible on pertubative level for all sufficiently excited states.

In section 7 we propose that in the thermodynamic limit large central charge 2d CFTs satisfy generalized eigenstate thermalization (GETH), with the values of qKdV charges forming a complete set of thermodynamically relevant quantities, which unambiguously determine expectation values of all local observables from the vacuum

family. Equivalence of ensembles further provides that local properties of an eigenstate can be described by the Generalized Gibbs Ensemble that only includes qKdV charges. In the case of a general initial state, upon equilibration, emerging Generalized Gibbs Ensemble will necessary include negative chemical potentials and holographically will be described by a quasi-classical black hole with quantum soft hair. In section 7.1 we define the regime of "quasi-classical gravity" in which we test generalized eigenstate thermalization. In section 7.2 we show in details how to explicitly check the hypothesis and illustrate it with level 6 quasi-primary example. In section 7.3 we list all the examples with analytic checks of GETH. In section 7.4 we show that among the states from the micro-canonical ensemble satisfying a simple scaling property an exponential majority of states satisfy "quasi-classical gravity" regime. Furthermore, most CFT eigenstates satisfy "quasi-classical gravity" regime. In section 7.5 we discuss the existence of chemical potentials which satisfy Generalized ETH and show that at least in some cases some of the chemical potentials should be negative.

We conclude in section 8 and outline some open problems for the future.

2 Preliminaries: classical KdV hierarchy

In this section we give a concise presentation of certain aspects of classical KdV hierarchy. In sections 2.1, 2.2 and 2.3 we mostly follow [9]. We start with the classic definition of KdV charges as generators of isospectral deformation for Sturm-Liouville equation. Then we introduce finite-zone "Novikov" potentials and develop perturbative technique for solving generalized KdV equations. Finally we obtain perturbative expression for KdV charges Q_{2k-1} in terms of action variables I_i. We will directly use these results in the next section where we will discuss spectrum for quantum KdV charges.

2.1 KdV hierarchy

Consider a Schrödinger eigenvalue problem (Sturm-Liouville equation),

$$-\psi'' + \frac{u}{4}\psi = \lambda\,\psi, \tag{2.1}$$

with periodic real-valued potential $u(\varphi) = u(\varphi+2\pi)$. The (non-degenerate) eigenvalues of periodic $\psi(2\pi) = \psi(0)$ and anti-periodic $\psi(2\pi) = -\psi(0)$ problems constitute the so-called spectral data of $u(\varphi)$. Different potentials may share the same spectral data. In fact there is an infinite family of infinitesimal deformations which are isospectral, i.e. preserve the spectral data. The isospectral deformations are generated by the Hamiltonian flow

$$\delta u = \frac{c}{24}\{Q_{2k-1}, u\}, \tag{2.2}$$

associated with the Poisson bracket [10],

$$\frac{c}{24}\{u(\varphi_1), u(\varphi_2)\} = -2\pi\mathcal{D}\delta(\varphi_1 - \varphi_2), \tag{2.3}$$

where c is some numerical parameter and differential operator $\mathcal{D}$ is defined as

$$\mathcal{D} = (\partial u) + 2u\partial - 2\,\partial^3. \tag{2.4}$$

The quantities Q_{2k-1} are the so-called KdV charges (or generators), which can be defined iteratively,

$$Q_{2k-1} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, R_k, \qquad \partial R_{k+1} = \frac{k+1}{2k-1}\mathcal{D}R_k, \tag{2.5}$$

$$R_0 = 1, \qquad \mathcal{D} \equiv (\partial u) + 2u\partial - 2\,\partial^3. \tag{2.6}$$

The polynomials R_k are called Gelfand-Dikii polynomials. They satisfy various relations [11], in particular

$$\frac{c}{24}\{Q_{2k-1}, u\} = (2k-1)\partial R_k. \tag{2.7}$$

First few R_k and Q_{2k-1} are given by

$$R_0 = 1, \quad R_1 = u, \quad R_2 = u^2 - \frac{4}{3}\partial^2 u, \quad R_3 = u^3 - 4u\partial^2 u - 2(\partial u)^2 + \frac{8}{5}\partial^4 u, \tag{2.8}$$

$$q_0 = u, \quad q_1 = u^2, \quad q_2 = u^3 + 2(\partial u)^2, \quad q_3 = u^4 + 8u(\partial u)^2 + \frac{16(\partial^2 u)^2}{5}, \tag{2.9}$$

where $2\pi\, Q_{2k-1} = \int_0^{2\pi} d\varphi\, q_k$. The quantities q_k and R_{k+1} differ only by a full derivative.

The name KdV comes from the form of the flow generated by Q_3. Assuming it defines a t-dependent function $u(t, x)$ via

$$\dot{u} = \frac{c}{24}\{Q_3, u\} = 6uu' - 4u''', \tag{2.10}$$

one can obtain the classic KdV equation. Generally speaking, the expression

$$\dot{u} = \frac{c}{24}\{Q_{2k-1}, u\} \tag{2.11}$$

defines all higher order KdV equations.

The KdV charges are in involution, $\{Q_{2k-1}, Q_{2l-1}\} = 0$, yet the action of Q_{2k-1} on a given $u(x)$ is usually non-trivial. It is known that the corresponding Hamiltonian flows exhaust all possible isospectral deformations.

2.2 Finite-zone "Novikov" solutions

For an arbitrary complex λ equation (2.1) has two solutions, which can be combined into a complex-valued quasi-periodic function ψ,

$$\psi(\varphi + 2\pi) = \psi(\varphi)e^{2\pi i p(\lambda)}. \tag{2.12}$$

For real λ the so-called quasi-momentum $p(\lambda)$ can be either real or pure imaginary. Values of $\lambda \in \mathbb{R}$ for which quasi-momentum $p(\lambda)$ is imaginary belong to the so-called forbidden zone. Forbidden zones stretch between two consecutive eigenvalues of periodic or anti-periodic problem. The zone disappears if the periodic or anti-periodic problem is double degenerate. The quasi-momentum is a complex function with the branch-cuts along the forbidden zones and $\lambda \leq \lambda_0$, where λ_0 is the energy of the ground state. For example all eigenvalue of the periodic and antiperiodic problems for the constant potential $u = u_0$ are double degenerate (except for the ground state),

$$\lambda^{(n)} = \frac{n^2 + u_0}{4}, \qquad n \geq 0. \tag{2.13}$$

Therefore there are no forbidden zones and $p(\lambda) = \sqrt{\lambda - u_0/4}$.

A special class of potentials with only a finite number of degeneracies lifted, and hence only a finite number of forbidden zones, are called finite-zone potentials. For example a one-zone potential will have a double-degenerate eigenvalue $\lambda^{(k)}$ for some $k \geq 1$ split into two, $\lambda_-^{(k)} = \lambda_1$ and $\lambda_+^{(k)} = \lambda_2$, while all other eigenvalues of periodic and antiperiodic problems remain double-degenerate (although their values are no longer given by (2.13)). It turns out that the values of all double-degenerate eigenvalues are uniquely fixed by the vacuum energy λ_0 and the ends of the zones, which in our case are λ_1, λ_2. Corresponding $p(\lambda)$ has two branch-cuts from $-\infty$ to λ_0 and from λ_1 to λ_2 and is given by an Elliptic integral discussed below. More generally a finite zone potential is specified by λ_i, $0 \leq i \leq 2n$ and is defined on a hyperelliptic curve of genus n.

Finite-zone potentials emerge as solutions of the static generalized KdV equation [12],

$$\{\mathcal{H}, u\} = 0, \qquad \mathcal{H} = \sum_{i=0}^{n} \mu_{2i+1} Q_{2i+1}, \qquad \mu_{2n+1} \neq 0. \tag{2.14}$$

In fact the following is true. Any solution of (2.14) is a $m \leq n$-zone potential, and all n-zone potentials can be obtained from (2.14) with the appropriate μ_{2i+1}.

For the given spectral data specified by the ends of the zones λ_i, $0 \leq i \leq 2n$, quasi-momentum is specified indirectly by its differential

$$dp = \frac{\lambda^n + r_{n-1}\lambda^{n-1} + \ldots r_0}{2y} d\lambda, \tag{2.15}$$

which is defined on the Riemann curve

$$y^2 = \prod_{i=0}^{2n} (\lambda - \lambda_i). \tag{2.16}$$

Coefficients r_i are fixed by the condition that

$$\oint_{a_i} dp = 2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} dp = 0, \tag{2.17}$$

vanishes for any a-cycle, defined as the brunch-cuts of y from λ_{2i-1} to λ_{2i}. Because of (2.17), function $p(\lambda)$ defined such that $dp = (\partial p/\partial\lambda)d\lambda$ is a well defined function on the Riemann curve (2.16). Furthermore for the potential associated with $\{\lambda_i\}$ to be 2π-periodic one must also require integrals over b-cycles

$$B_i = \oint_{b_i} dp = 2 \int_{\lambda_{2i-2}}^{\lambda_{2i-1}} dp \tag{2.18}$$

to be integer-valued

$$B_i = k_i - k_{i-1} \tag{2.19}$$

As a complex function $p(\lambda)$ has branch-cuts along the forbidden zones, and therefore finite zone solutions are also called finite- or multi-cut solutions.

Each Hamiltonian flow generated by Q_{2k-1} is isospectral, hence it deforms a finite-zone solution into another, such that involution condition $\{\mathcal{H}, u\} = 0$ continues being satisfied. For any fixed Q_{2k-1}, $1 \le k \le n+1$, values of all higher charges Q_{2k-1}, $k > n+1$ are fixed and the space of solutions is an n-dimensional torus (Jacobian of the hyperelliptic curve (2.16)). All charges Q_{2k-1} generate a flow on the Jacobian, which is ergodic in a general case. The exception being the flow generated by Q_1 which is equivalent to the shift $\varphi \to \varphi + \text{const}$ and therefore 2π-periodic.

2.3 Example: one-cut solutions

In this section the exact solution of the generalized KdV equation (2.14) for $n = 1$ is presented. We mostly follow [9]. For $n = 1$ the equation (2.14) takes the form

$$\frac{c}{24}\{Q_3 + \alpha\, Q_1, u\} = 6u\, u' - 4u''' + \alpha\, u' = 0. \tag{2.20}$$

By integrating this equation twice the problem can be reduced to of a particle moving in a cubic potential,

$$\frac{u'^2}{2} + V(u) = E, \quad 2(E - V) = \frac{1}{2}(u - u_1)(u - u_2)(u - u_3), \tag{2.21}$$

$$s_1 = u_1 + u_2 + u_3 = -\alpha/2. \tag{2.22}$$

Two out of three parameters u_i are free. They specify the values of Q_1, Q_3 evaluated on the solution. From (2.21) $u(\varphi)$ can be obtained in terms of Weierstrass's elliptic function $\wp$ by specifying the initial condition $u(\varphi_0)$. The solution is to be 2π-periodic, which imposes a condition on "energy" E, leaving only one free parameter, besides φ_0. There is in fact an infinite tower of solutions with the period $2\pi/k$ for positive integer k, each being parametrized by one continuous parameter, in addition to φ_0. Weierstrass's function is associated with a torus, and it is convenient to choose s_1 and torus modular parameter $q = e^{i\pi\tau}$ as the independent parameters of one-cut solution,

$$u_1 = \frac{s_1}{3} - \frac{2k^2}{3}\left(\theta_2(0;q)^4 + \theta_3(0;q)^4\right), \tag{2.23}$$

$$u_2 = \frac{s_1}{3} + \frac{2k^2}{3}\left(\theta_2(0;q)^4 - \theta_4(0;q)^4\right), \tag{2.24}$$

$$u_3 = \frac{s_1}{3} + \frac{2k^2}{3}\left(\theta_3(0;q)^4 + \theta_4(0;q)^4\right). \tag{2.25}$$

The order $u_1 \le u_2 \le u_3$ is chosen in such way, that the periodic solution describes the oscillations of a "particle" between u_1 and u_2. Sometimes instead of u_i it is convenient to use s_1 and

$$s_2 = u_1u_2 + u_2u_3 + u_1u_3 = -\frac{k^4}{\pi^4}g_2(\tau) + \frac{s_1^2}{3}, \tag{2.26}$$

$$s_3 = u_1u_2u_3 = -\frac{2k^6}{\pi^6}g_3(\tau) - \frac{k^4}{3\pi^4}s_1g_2(\tau) + \frac{s_1^3}{27}. \tag{2.27}$$

Here g_2 and g_3 are modular forms. The value of Q_1 can be written in terms of u_i as

$$Q_1 = u_3 + (u_2 - u_3)\frac{{}_2F_1\left(\frac{3}{2},\frac{1}{2},1;\frac{u_2-u_1}{u_3-u_1}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;\frac{u_2-u_1}{u_3-u_1}\right)}. \tag{2.28}$$

Higher KdV charges can be expressed through Q_1 and s_i,

$$Q_3 = \frac{1}{3}(2s_1Q_1 - s_2), \quad Q_5 = \frac{2s_1^2 - s_2}{5}Q_1 - \frac{s_1s_2 + s_3}{5}. \tag{2.29}$$

In terms of the spectral data, one-zone potential is characterized by three eigenvalues of the Schrödinger equation (2.1), the ground state λ_0, and the ends of the forbidden zone, λ_1, λ_2,

$$\lambda_0 = \frac{u_1 + u_2}{8}, \quad \lambda_1 = \frac{u_1 + u_3}{8}, \quad \lambda_2 = \frac{u_2 + u_3}{8}. \tag{2.30}$$

When the "energy" E is small, meaning $E - V$ approaches zero, the "particle" oscillates near the local minimum of the potential with the period $2\pi/k$, and the values of λ_1

and λ_2 approach $\lambda_0 + k^2/4$ from both sides. This corresponds to a small perturbation of the constant potential which removes degeneracy of just one eigenvalue in (2.13).

Besides one-cut solutions with non-constant $u(\varphi)$ there are also two φ-independent solutions of (2.21) corresponding to a "particle" sitting at the top or bottom of the potential.

2.4 Perturbative approach: "opening" a zone

Let us now consider the potential $u = u_0 + \epsilon \cos(k\varphi) + O(\epsilon^2)$ where u_0 is a constant, ℓ is positive integer, and ϵ is some infinitesimal parameter. Using quantum mechanics perturbation theory we find at leading order that all eigenvalues of periodic and anti-periodic problems remain the same and double-degenerate, except for λ_ℓ which splits into

$$\lambda_k^{\pm} = u_0 + \frac{k^2}{4} \pm \frac{\epsilon}{2}. \tag{2.31}$$

Hence now there are two forbidden zones, $(-\infty, u_0)$ and $(\lambda_k^-, \lambda_k^+)$.

Finite-zone potentials are characterized by the ends of non-degenerate zones λ_i. For the zero-zone potential above there is only one parameter $\lambda_0 = u_0$. After one zone is opened, there are three parameters $\lambda_0 = u_0$, $\lambda_1 = \lambda_k^-$ and $\lambda_2 = \lambda_k^+$.

A given set $\{\lambda_i\}$ which satisfies (2.17), (2.19), such that only $n+1$ parameters are independent, defines periodic potential $u(\varphi)$, but in a non-unique way. Individual potentials are labeled by points of the Jacobian of curve (2.16), with all of them sharing the same spectrum. In other words isospectral potentials form an n-dimensional torus, while full space of n-zone potentials is therefore $2n+1$ dimensional.

We now consider a space of all n-zone potentials sharing the same h. This is a $2n$-dimensional subspace within the orbit parametrized by h, which we will denote as $\mathcal{F}_n(h)$. The pullback of symplectic form on this space is non-degenerate, hence it is also a symplectic manifold equipped with Poisson bracket. Isospectral flows leave this manifold invariant. Upon restricting to $\mathcal{F}_n(h)$, only first n KdV Hamiltonians remain algebraically independent. The flows they generate move u along the Jacobian of (2.16), which is the Liouvillian torus of a complete integrable dynamical system defined by Q_{2k-1}, $k \leq n$. In other words the geometry of $\mathcal{F}_n(h)$ is a n-dimensional torus parametrized by angle variables fibered above a base parametrized by n variables Q_{2k-1}. Alternatively, one can introduce n action variables I_i parameterizing the base and forming canonical conjugate pars with angle variables.

In terms of dp (2.15) values of KdV charges are given by an expansion at infinity [12]

$$Q_{2k-1} = \frac{2\Gamma(k+1)\Gamma(1/2)}{\Gamma(k+1/2)} \frac{1}{2\pi i} \oint\limits_{\infty} dp\, \lambda^{k-1/2}, \tag{2.32}$$

while the action variables are

$$I_i = -\frac{1}{2\pi} \oint\limits_{a_i} p \frac{d\lambda}{\lambda} = \frac{1}{2\pi} \oint\limits_{a_i} dp \ln \lambda. \tag{2.33}$$

Functional dependence of Q_{2k-1} for $k > n$ on charges Q_{2k-1} for $k \leqslant n$ readily follows from (2.32) and the form of dp (2.15).

Our task is conceptually trivial: we want to learn an explicit change of variables on the base of $\mathcal{F}_n(h)$ from Q_{2k-1} to I_i. In practice explicit expressions for $Q_{2k-1}(h, I_i)$ is not available in the closed form, we therefore will find it in first few orders by expanding in powers of I_i. There is one notable exception, using Riemann bilinear relation one can show in full generality

$$Q_1 = h + \sum_i i\, I_i, \tag{2.34}$$

where h is related to the co-adjoint invariant – quasi-momentum at zero [12],

$$p(0)^2 = -h. \tag{2.35}$$

2.5 One-zone potentials

Before we consider one-zone potential in detail, we revisit zero-zone potential $u = u_0 \equiv \lambda_0$ and readily find differential

$$dp = \frac{d\lambda}{2\sqrt{\lambda - \lambda_0}} \tag{2.36}$$

to be defined on a Riemann sphere. This is the simplest possible case. In this case $p = \sqrt{\lambda - \lambda_0}$, $u(\varphi) = \lambda_0 = h$ and the whole symplectic space $\mathcal{F}_0(h)$ shrinks to a point. All KdV Hamitonians are fixed by h, $Q_{2k-1} = h^k$ with all action variables identically equal zero.

Next, we consider a differential

$$dp = \frac{(\lambda - r_0)d\lambda}{2\sqrt{(\lambda - \lambda_0)(\lambda - \lambda_1)(\lambda - \lambda_2)}} \tag{2.37}$$

parameterized by λ_i, r_0. It is defined on a torus – a Riemann curve of genus one. We assume that (λ_2, λ_1) correspond to k-th zone. After satisfying (2.17) and (2.19) as well as the orbit constraint $-p(0)^2 = h$ we find one-parametric family

$$\lambda_2 - \lambda_1 = \frac{k^2}{2}\theta_2(\tau)^4, \quad \lambda_2 - \lambda_0 = \frac{k^2}{2}\theta_3(\tau)^4, \quad \lambda_1 - \lambda_0 = \frac{k^2}{2}\theta_4(\tau)^4. \tag{2.38}$$

Here τ is purely imaginary and $\mathrm{Im}(\tau) > 0$. The orbit constraint can be rewritten in the following form

$$\sqrt{h} = \int_0^{\lambda_0} d\lambda \frac{\lambda - r}{\sqrt{y^2}}. \tag{2.39}$$

After introducing $q = e^{2\pi i \tau}$, we obtain the following expansion

$$\begin{aligned} I_k =& \frac{32k^3q^2}{h+k^2} + \frac{64q^4\left(3h^2k^3 + 12hk^5 + k^7\right)}{\left(h+k^2\right)^3} + \frac{128k^3q^6\left(3h^4 + 42h^3k^2 + 108h^2k^4 - 58hk^6 + k^8\right)}{\left(h+k^2\right)^5} \\ &+ \frac{128k^3q^8\left(7h^6 + 156h^5k^2 + 1083h^4k^4 + 1232h^3k^6 - 4035h^2k^8 + 788hk^{10} + k^{12}\right)}{\left(h+k^2\right)^7} + \mathcal{O}(q^{10}). \end{aligned}$$

Solving it inversely, we obtain

$$\begin{aligned} q^2 =& \frac{(h+k^2)}{32k^3} I_k - \frac{\left(3h^2 + 12hk^2 + k^4\right)}{512k^6} I_k^2 + \frac{\left(15h^3 + 87h^2k^2 + 105hk^4 + k^6\right)}{8192k^9} I_k^3 \\ &- \frac{\left(187h^4 + 1402h^3k^2 + 3012h^2k^4 + 1606hk^6 + k^8\right)}{262144k^{12}} I_k^4 + \mathcal{O}(I_k^5). \end{aligned}$$

Using this expression we find

$$Q_{2n-1} = h^n + f_k^{(n,1)} I_k + f_k^{(n,2)} I_k^2 + \mathcal{O}(I^3), \tag{2.40}$$

where

$$f_k^{(n,1)} = \frac{(2n)!!k(h+k^2)^{n-1}}{2(2n-3)!!} \sum_{m=0}^{n-1} \frac{(2m-1)!!}{(2m)!!} \left(\frac{h}{h+k^2}\right)^m, \tag{2.41}$$

$$f_k^{(n,2)} = -\frac{(2n)!!(h+k^2)^{n-2}}{16(2n-3)!!} \sum_{m=0}^{n-1} (3h + k^2 - 4k^2 m) \sum_{j=0}^{m-1} \frac{(2j-1)!!}{(2j)!!} \left(\frac{h}{h+k^2}\right)^j. \tag{2.42}$$

2.6 Two-zone potentials

We parametrize two-zone coadjoint orbit as follows

$$\lambda_0 = \frac{u_0}{4} + b_{01}\epsilon_1^2 + b_{02}\epsilon_2^2 + b_{12}\epsilon_1\epsilon_2, \tag{2.43}$$

$$\lambda_1 = u_0/4 + k^2/4 - \epsilon_1 + b_1\epsilon_1^2, \quad \lambda_2 = u_0/4 + k^2/4 + \epsilon_1 + b_1\epsilon_1^2, \tag{2.44}$$

$$\lambda_3 = u_0/4 + l^2/4 - \epsilon_2 + b_2\epsilon_2^2, \quad \lambda_4 = u_0/4 + l^2/4 + \epsilon_2 + b_2\epsilon_2^2. \tag{2.45}$$

This parametrization includes five parameters b_i which can be determined via two a-cycle conditions, two b-cycle conditions and the condition (2.34). You find more details in [13].

After the dust settles, we obtain

$$Q_{2n-1} = h^n + \sum_k f_k^{(n,1)} I_k + f_k^{(n,2)} I_k^2 + \frac{1}{2} \sum_{k \neq \ell} f_{k,\ell}^{(n)} I_k I_\ell + \mathcal{O}(I^3) \tag{2.46}$$

with $f_k^{(n,1)}, f_k^{(n,2)}$ were given in the previous subsection and

$$f_{k,\ell}^{(n)} = \frac{(2n)!!k\ell}{4(2n-3)!!(k^2-\ell^2)} \sum_{m=0}^{n-1} \sum_{j=0}^{m-1} \frac{(2j-1)!!h^j}{(2j)!!} \left[(h+k^2)^{n-j-1} - (h+\ell^2)^{n-j-1} \right] \tag{2.47}$$

$$+ \frac{(2n)!!k\ell}{4(2n-3)!!} \sum_{m=0}^{n-1} \sum_{j=0}^{m-1} \frac{m(2j-1)!!h^j}{(2j)!!} \left[(h+k^2)^{m-j-1}(h+\ell^2)^{n-m-1} + (h+\ell^2)^{m-j-1}(h+k^2)^{n-m-1} \right].$$

The same expression can be rewritten as follows

$$\begin{aligned} Q_{2n-1} = h^n &+ \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} h^{n-1-j} k^{2j+1} I_k \\ &- \sum_k \sum_{m=0}^{n-1} \frac{(2n-1)(2mn-2n+m+2)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{16\,\Gamma\left(m+\frac{3}{2}\right)\Gamma(n-m)} h^{n-m-1} k^{2m} I_k^2 \\ &+ \frac{1}{2} \sum_{k,\ell} \sum_{m=1}^{n-1} \frac{(2n-1)^2\Gamma(n+1)\Gamma\left(\frac{1}{2}\right) h^{n-m-1}}{2^2\Gamma(n-m)\Gamma\left(m+\frac{3}{2}\right)} \sum_{s=0}^{m-1} k^{2(m-s)-1} \ell^{2s+1} I_k I_\ell + \mathcal{O}(I^3) \end{aligned} \tag{2.48}$$

As the result of this section we get a perturbative expression for classical KdV charges Q_{2n-1} in terms of action variables I_i and orbit invariant h. In the next section we will show how to "quantize" this expression in order to establish quantum version of the statement (2.48).

3 Quantum KdV hierarchy and 2d CFT

In this section we show how perturbative approach from the previous section for classical KdV charges helps to establish the spectrum of quantum KdV charges as a series expansion for large central charge c. As the result we provide an explicit expression for the spectrum of all quantum KdV in large central charge c limit up to a third order of expansion.

Section 3.1 is devoted to well-known connection between co-adjoint orbits of Virasoro algebra and periodic potentails $u(\varphi)$. This presentation mostly follows [9]. Then,

we introduce an infinite tower of quantum KdV charges $\hat{Q}_{2k-1}$ for 2d CFTs. In section 3.4 we compute eigenvalues for $\hat{Q}_{2k-1}$ on (highest weight) primary fields via ODE/IM correspondence equation. Finally, we combine these results in section 3.5 and obtain spectrum for quantum KdV charges in large c limit.

3.1 Co-adjoint orbit of Virasoro algebra

Consider the group diff $\mathbb{S}^1$ of diffeomorphisms of a circle. Elements of diff $\mathbb{S}^1$ are monotonically increasing functions $\tilde{\varphi} = g(\varphi)$,

$$g(2\pi) = g(0) + 2\pi, \tag{3.1}$$

such that g is an invertible map of a circle into itself, $g(\varphi) = g(\varphi') \Rightarrow \varphi = \varphi'$. Corresponding Lie algebra is the Witt algebra of vector fields on a circle $f(\varphi)\partial_\varphi$.

It is useful to consider an action of this group on our periodic potential $u(\varphi)$, $u(\varphi + 2\pi) = u(\varphi)$ and a wave-function $\psi(\varphi)$ satisfying Schrödinger (or Hill's) equation,

$$-\psi'' + \frac{u}{4}\,\psi = 0. \tag{3.2}$$

Note, that equation (3.2) corresponds to (2.1) for $\lambda = 0$. So, the setup is not exactly the same as in previous section. Diffeomorphisms $g \in$ diff $\mathbb{S}^1$ naturally act on u and ψ,

$$g: \ \psi(\varphi) \to \tilde{\psi}(\tilde{\varphi}), \tag{3.3}$$

$$g: \ u(\varphi) \to \tilde{u}(\tilde{\varphi}), \tag{3.4}$$

such that the Hill's equation continue being satisfied (the derivative is with respect to $\tilde{\varphi}$),

$$-\tilde{\psi}'' + \frac{\tilde{u}}{4}\,\tilde{\psi} = 0. \tag{3.5}$$

The new potential and the new wave-function are defined as

$$\tilde{\psi}(\tilde{\varphi}(\varphi)) = \psi(\varphi)\left(\frac{d\tilde{\varphi}}{d\varphi}\right)^{1/2}, \tag{3.6}$$

$$\tilde{u}(\tilde{\varphi}(\varphi)) = \left(\frac{d\tilde{\varphi}}{d\varphi}\right)^{-2}\left[u(\varphi) + 2\{\tilde{\varphi}, \varphi\}\right]. \tag{3.7}$$

where for any $\theta(\varphi)$

$$\{\theta, \varphi\} \equiv \frac{\theta'''}{\theta'} - \frac{3}{2}\left(\frac{\theta''}{\theta'}\right)^2, \tag{3.8}$$

defines the Schwarzian derivative. An infinitesimal transformation

$$g(\varphi) = \varphi - \epsilon f(\varphi) \tag{3.9}$$

acts on the potential as follows,

$$\tilde{u}(\varphi) = u(\varphi) + \epsilon\, \mathcal{D} f, \qquad \mathcal{D} \equiv (\partial u) + 2u\partial - 2\,\partial^3. \tag{3.10}$$

It is well known that this action is exactly the the action of Virasoro algebra, central extension of Witt algebra, on its co-adjoint orbit.

Elements of Virasoro algebra are the pairs (f, a) where f is a vector field and a is a $\mathbb{C}$-number with the following commutation relation

$$[(f_1, a_1), (f_2, a_2)] = (f_1 f_2' - f_1' f_2, a), \qquad a = \int_0^{2\pi} d\varphi (f_1''' f_2 - f_1 f_2''') . \tag{3.11}$$

Co-adjoint space is the linear space dual to the algebra. Its elements are the pairs $[u, \hat{t}]$ where $u(\varphi)d\varphi^2$ is a "two-differential" and $[0, \hat{t}]$ is an element formally dual to $(0, 1)$. Considering $\hat{t}$ to be common for all elements one can reduce bulky notation $[u, \hat{t}]$ to simple u, such that the scalar product is

$$\langle (f, a), u \rangle = a + \int_0^{2\pi} d\varphi\, u f. \tag{3.12}$$

It is easy to see that $\langle (f, a), u \rangle$ is invariant under the action of a Virasoro algebra element (v, b) provided,

$$\delta f = v f' - v' f, \quad \delta a = \int_0^{2\pi} d\varphi (v''' f - v f'''), \quad \delta u = \mathcal{D} v. \tag{3.13}$$

Action of $\mathrm{diff}\,\mathbb{S}^1$ (3.13) foliates the space of all $u(\varphi)$ into orbits – the co-adjoint orbits of Virasoro algebra. Starting with some potential u one defines a sub-algebra of stabilizers f of u such that

$$\delta u = \mathcal{D} f = 0. \tag{3.14}$$

In full generality there could be either one or three linearly independent stabilizers [14], which must be closed in the Lie algebra sense. Then the orbit is defined by the action of all possible diffeomorphisms $g(\varphi)$ on the given u, modulo the stabilizer subgroup. The simplest orbit is obtained starting from a constant $u(\varphi) = u_0$. In this case the stabilizer is unique, $f = 1$, up to an overall rescaling, with an exception of the case when $u_0 \neq -n^2$ for some integer n. These are the orbits $\mathrm{diff}\,\mathbb{S}^1/\mathbb{S}^1$ [14]. Quantization of such an orbit gives rise to the corresponding Verma module.

When $u(\varphi)$ belongs to an orbit $\mathrm{diff}\,\mathbb{S}^1/\mathbb{S}^1$ the stabilizer vector field f for each u is unique and sign-definite. The converse is also correct and easy to see. Considering f, such that it is sign-definite and $\mathcal{D}f = 0$, one can notice that

$$2\pi f_0^{-1} = \int_0^{2\pi} \frac{d\varphi}{f}, \tag{3.15}$$

is invariant under the diffeomorphism $\mathrm{diff}\,\mathbb{S}^1$, as follows straightforwardly from (3.11). Next, one can define the diffeomorphism $\varphi \to \tilde{\varphi} = g(\varphi)$,

$$d\tilde{\varphi} = f_0 \frac{d\varphi}{f}, \tag{3.16}$$

which brings f to a constant form f_0. This is the diffeomorphism which brings $u(\varphi)$ to a constant, as follows from applying (3.7),

$$\tilde{u}(\tilde{\varphi}) = u_0 = \frac{uf^2 + f'^2 - 2ff''}{f_0^2}. \tag{3.17}$$

u_0 defined this way is a constant, which can be verified by differentiating it, $\tilde{u}' = (f/f_0)\partial_\varphi u_0 = (f^2/f_0^3)\mathcal{D}f = 0$. An alternative way to obtain the same expression is to start with $\mathcal{D}f = 0$ and solve it as an equation for u,

$$u(\varphi) = \frac{u_0 f_0^2 - f'^2 + 2ff''}{f^2}. \tag{3.18}$$

Here $u_0 f_0^2$ appears as an integration constant. It is straightforward to see that (3.18) is compatible with (3.13) only if $u_0 f_0^2$ is invariant under the diffeomorphisms. Hence $u_0 f_0^2$ is equal to uf^2 when $u(\varphi)$ is φ-independent and hence so is f. Finally, u_0 is the only invariant characterizing the orbit, and its invariance under the diffeomorphisms follows straightforwardly from (3.17) and (3.11).

The space of all potentials is a Poisson manifold with the Poisson bracket (2.3), Written in terms of the Fourier series

$$\frac{c}{24}(u(\varphi) + 1) = \sum_k \ell_k e^{ik\varphi}, \tag{3.19}$$

the Poisson brackets (2.3) reduce to Virasoro algebra

$$i\{\ell_n, \ell_m\} = (n-m)\ell_{n+m} + \frac{c(n^3 - n)}{12}\delta_{n+m}. \tag{3.20}$$

In particular for any functional $\mathcal{H}[u(\varphi)]$,

$$\frac{c}{24}\{\mathcal{H}, u(x)\} = \mathcal{D}f, \qquad f = 2\pi \frac{\delta \mathcal{H}}{\delta u(x)}. \tag{3.21}$$

Here $\mathcal{D}f$ is as the Hamiltonian vector field associated with $\mathcal{H}$ in the space of potentials $u(\varphi)$.

Since the Hamiltonian vector field (3.21) has the form of (3.13) with some appropriate $v = f$, Hamiltonian flow does not move $u(x)$ away from the orbit, hence on the space of all potentials the Poisson bracket is degenerate. Restricting it to a particular orbit removes this degeneracy, and (2.3) defines a symplectic form, such that each orbit is a symplectic manifold. This symplectic form is the Kirillov-Kostant form on the co-adjoint orbit of Virasoro algebra [15], as is also evident from (3.20).

3.2 qKdV symmetry in CFT$_2$

The co-adjoint orbit of Virasoro algebra $\mathrm{diff}\,\mathbb{S}^1/\mathbb{S}^1$ is a symplectic manifold with the non-degenerate Poisson bracket (2.3). Upon quantization, it gives rise to Verma module with the primary (highest weight) state $|\Delta\rangle$ of dimension

$$\Delta = \frac{c}{24}(u_0 + 1). \tag{3.22}$$

Upon quantization Fourier modes ℓ_n (3.19) of the classical field $u(\varphi)$ become Virasoro algebra generators L_n, while u itself becomes stress-energy tensor in a CFT$_2$ on a cylinder with the central charge c,

$$T = \frac{c}{24}u. \tag{3.23}$$

It is then easy to recognize (3.7) as the standard expression for the change of stress-energy tensor upon a coordinate transformation.

The Poisson brackets (2.3) were originally introduced in the context of higher KdV equations [10], and soon the connection with the Virasoro algebra was noticed by Gervais and Neveu [15]. Later Gervais suggested that classical KdV charges (2.5), upon quantization, should give rise to mutually-commuting quantum operators [16, 17]. While being very intuitive, this proposal is not trivial. Since the higher generators are non-linear in u, their quantum counterparts will depend on the normal ordering and may no longer commute as a result. This question was fully resolved only in [18–20] where existence of an infinite tower of local commuting qKdV charges $\hat{Q}_{2k-1}$ was established. Their definition, besides normal ordering, is also explicitly c-dependent. For example, first few charges in terms of the stress-tensor are

$$\hat{Q}_1 = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, T, \quad \hat{Q}_3 = \frac{1}{2\pi}\int_0^{2\pi} d\varphi (TT), \quad \hat{Q}_5 = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, (T(TT)) + \frac{c+2}{12}(\partial T)^2.$$

In our notations classical charges Q_{2k-1} give rise to $\left(\frac{c}{24}\right)^{-k-1}\hat{Q}_{2k-1}$. Notice however that $\left(\frac{c}{24}\right)^3 Q_5$ is *not* equal to $\hat{Q}_5$ upon substitution $u \to \frac{24}{c}T$ and normal ordering. An extra term $(\partial T)^2/6$ is necessary to assure commutativity with $\hat{Q}_3$.

In terms of Virasoro algebra generators $\hat{Q}_1$ is simply the CFT Hamiltonian $L_0 - \frac{c}{24}$. Expressions for $\hat{Q}_3, \hat{Q}_5$ in terms of L_n were obtained in [18], as well as for $\hat{Q}_7$ in [21]. Generally, explicit expressions for $\hat{Q}_{2k-1}$ are not known neither in terms of Virasoro generators L_n, nor as integrals over φ.

3.3 2d CFT states

The first qKdV charge Q_1 plays role of the Hamiltonian in 2d CFT,

$$Q_1 = L_0 - \frac{c}{24}. \tag{3.24}$$

This Hamiltonian is highly degenerate. All Hilbert states in terms of eigenstates of Q_1 can be enumerated as follows,

$$|E\rangle = |m_i, \Delta\rangle = L_{-m_1} \dots L_{-m_k} |\Delta\rangle, \qquad m_i \geq m_j \quad \text{for} \quad i \geq j, \quad \sum m_i = n. \tag{3.25}$$

At each level n there are $p(n)$ (number of integer partitions of n) eigenstates of Q_1 sharing the same energy $E = \Delta + n - \frac{c}{24}$. Taking into account all qKdV charges Q_{2k-1} in principle, we can construct a unique basis in which all these charges are simultaneously diagonalized, since all Q_{2k-1} commute. The importance of this basis will be evident when we will discuss Eigenstate Thermalization. There are no known way to explicitly construct such basis, however, as we will see in section 4.1, similar to (3.25) we can parametrize this basis by natural numbers. These natural numbers can be conveniently combined into a Young tableau. We will use so-called "boson representation" $n_i \geqslant 0, \quad i = 1, 2, ...$ which counts the number of rows of length i in a given Young tableau,

$$|n_i\rangle \equiv |n_1, ...\rangle\,, \quad \sum_{i=1}^{\infty} i n_i = n. \tag{3.26}$$

Corresponding eigenvalues of Q_{2k-1} we will denote as λ_{2k-1},

$$Q_{2k-1}\,|n_i\rangle = \lambda_{2k-1}\,|n_i\rangle\,. \tag{3.27}$$

3.4 "Energies" of primary states

In section 2.6 we found classical expression for Q_{2k-1} in term of action variables I_i and h which we now can interpret as the orbit invariant. To quantize this result, at least naively, I_i should be promoted to an integer boson occupation number, while h would become dimension of the highest weight (primary) state Δ, marking representation of the Virasoro algebra. It is easy to see this naive recipe fails already for the values of

$\hat{Q}_{2k-1}$ on primary state Δ. Indeed, taking all I_i to zero, we readily find $Q_{2k-1} = h^k$, which upon naive quantization yields $\lambda^0_{2k-1} = \Delta^k$ where

$$\hat{Q}_{2k-1}|\Delta\rangle = \lambda^0_{2k-1}|\Delta\rangle. \tag{3.28}$$

This answer is missing c-dependent terms. Explicit values of λ^0_{2k-1} for $k \leq 8$ were calculated in [22] using brute-force approach, using explicit expressions for $\hat{Q}_{2k-1}$ in terms of free field representation. The pattern is clear, while Δ^k is indeed a leading term, full expression is a polynomial in both Δ and c of order k.

There is no consistence recipe to obtain exact λ^0_{2k-1} from the quasi-classical quantization, hence our strategy will be the following. We will combine exact expression for λ^0_{2k-1} in the large c limit, which will be obtained in this section, with the classical result of section 2.6, to find spectrum of Q_{2k-1} for excited states in next section.

To find λ^0_{2k-1} we use ODE/IM correspondence, initiated in [23, 24] and more recently developed in [25], which relates qKdV spectrum to solutions of an auxiliary Schrodinger equation:

$$\partial_x^2\Psi(x) + \left(E - x^{2\alpha} - \frac{l(l+1)}{x^2}\right)\Psi(x) = 0, \tag{3.29}$$

where

$$(l+1/2)^2 = 4(\alpha+1)\tilde{\Delta}, \quad \tilde{c} = -\frac{\alpha^2}{4(\alpha+1)} \tag{3.30}$$

and

$$\tilde{c} = \frac{c-1}{24}, \quad \tilde{\Delta} = \Delta - \tilde{c}. \tag{3.31}$$

Taking $c \to \infty$ limit equation (3.29) can be solved using WKB approximation by systematically expanding in a small parameter. This leads to a quadratic ODE which can be solved iteratively. We skip all details (which you can find in [13]) and only write down iterative relation which defines coefficients $c_k^{(n)}$,

$$\sum_{j=0}^{n}\sum_{p=0}^{j}\sum_{q=0}^{n-j}\delta_{p+q,k}c_p^{(j)}c_q^{(n-j)} - 2\left[n-k-u-\frac{n-2}{2\alpha}\right]c_{k-1}^{(n-1)} + (2k-3n+4)\,c_k^{(n-1)} = 0, \tag{3.32}$$

for $n \geq 1$, we formally assumed $c_{-1}^{(n)} = c_{n+1}^{(n)} = 0$, $u^2 = -\tilde{\Delta}/\tilde{c}$, and the starting values are

$$c_0^{(0)} = -\frac{1}{\alpha}, \qquad c_0^{(1)} = -\frac{1}{2}, \quad c_1^{(1)} = \frac{1}{2\alpha} - u, \ldots \tag{3.33}$$

Coefficients $c_k^{(n)}$ determine values of Q_{2n-1} acting on primaries,

$$\lambda^0_{2n-1} = \frac{(2n-1)\Gamma(n+1)}{\sqrt{\pi}\Gamma(1-\frac{2n-1}{2\alpha})(4\alpha+4)^n}\sum_{k=0}^{2n} c_k^{(2n)}\Gamma\left(-\frac{3}{2}(2n-1)+k\right)\Gamma\left(2n-k-\frac{2n-1}{2\alpha}\right). \tag{3.34}$$

Although this is not obvious, λ^0_{2n-1} given by (3.34) is a polynomial in terms of $\tilde{\Delta}$ and $\tilde{c}$. After some algebra we find leading order expansion

$$\lambda^0_{2n-1} = \tilde{\Delta}^n + \sum_{j=0}^{n-1} R^{(1)}_{n,j}\tilde{\Delta}^{n-j-1}\,\tilde{c}^j + \sum_{j=0}^{n-2} R^{(2)}_{n,j}\tilde{\Delta}^{n-j-2}\,\tilde{c}^j + \mathcal{O}(\tilde{c}^{n-3}). \tag{3.35}$$

where

$$R^{(1)}_{n,j} = \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{4\Gamma(j+\frac{3}{2})\Gamma(n-j)}\zeta(-2j-1), \tag{3.36}$$

$$R^{(2)}_{n,j} = \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{96\Gamma(j+\frac{5}{2})\Gamma(n-j-1)} \tag{3.37}$$
$$\times\left\{-6\zeta(-2j-3)\left[2j+3-(2n-1)\sum_{\ell=0}^{j+1}\frac{1}{2\ell+1}\right]+3(2n-1)\sum_{s=0}^{j}\zeta(-2(j-s)-1)\,\zeta(-2s-1)\right\}.$$

3.5 Spectrum of quantum $\hat{Q}_{2k-1}$ generators

At this point we are ready to combine classical pertubative expression for $Q_{2n-1}(h, I_k)$ (2.48) with the "energies" of primary state (3.35) to obtain λ_{2n-1} up to first three orders in $1/\tilde{c}$ expansion. The naive quantization scheme maps classical to quantum charges as follows

$$Q_{2n-1}(h, I_k) \to \lambda_{2n-1} = \tilde{c}^n\, Q_{2n-1}(\tilde{\Delta}/\tilde{c}, (n_k+1/2)/\tilde{c}). \tag{3.38}$$

Infinite sums appearing because of the Maslov index $+1/2$ should be regularized using zeta-function regularization. It is now straightforward to see that we immediately reproduce leading $1/\tilde{c}$ term (3.40) as well as certain terms from (3.37). The full quantum

spectrum we present as follows

$$\begin{aligned}\lambda_{2n-1} &= \tilde{\Delta}^n + \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} \tilde{c}^j \tilde{\Delta}^{n-1-j} k^{2j+1} \tilde{n}_k \\ &- \frac{n(n-1)(2n-1)\tilde{\Delta}^{n-1}}{4c} \\ &+ \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{8\Gamma(j+\frac{3}{2})\Gamma(n-j)} \left(-2j + (2n-1) \sum_{\ell=0}^{j} \frac{1}{2\ell+1} - \right) \tilde{c}^{j-1} \tilde{\Delta}^{n-1-j} k^{2j+1} \tilde{n}_k \\ &- \sum_k \sum_{m=0}^{n-1} \frac{(2n-1)(2mn+2n-3m-2)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{16\,\Gamma\left(m+\frac{3}{2}\right)\Gamma(n-m)} \tilde{c}^{m-1} \tilde{\Delta}^{n-m-1} k^{2m} \tilde{n}_k^2 \\ &+ \frac{1}{2} \sum_{k,\ell} \sum_{m=1}^{n-1} \frac{(2n-1)^2\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2^2\Gamma(n-m)\Gamma\left(m+\frac{3}{2}\right)} \tilde{c}^{m-1} \tilde{\Delta}^{n-m-1} \sum_{s=0}^{m-1} k^{2(m-s)-1} \ell^{2s+1} \tilde{n}_k \tilde{n}_\ell \\ &+ \mathcal{O}(c^{n-3}).\end{aligned} \tag{3.39}$$

Here $\tilde{n}_k = n_k + 1/2$ and we added terms of order $\tilde{c}^{n-2}$ linear and zeroth order in n_k such that upon taking $n_k = 0$ we obtain λ_k^0. This expression is our final result, to which we give a consistency check in the next subsection using computer algebra.

In leading order the above result reduces to "free boson" expression

$$\begin{aligned}\lambda_{2k-1} &= \tilde{\Delta}^k + \sum_{p=0}^{k-1} \xi_k^p \, \tilde{\Delta}^{k-1-p} \, \tilde{c}^p \left(\sum_{i=1}^{\infty} i^{2p+1} n_i + \frac{\zeta(-2p-1)}{2} \right) + O(\tilde{c}^{k-2}), \\ \xi_k^p &= \frac{(2k-1)\Gamma(k+1)\Gamma(1/2)}{2\,\Gamma(p+3/2)\Gamma(k-p)}, \qquad \tilde{\Delta} = \Delta - \tilde{c}, \qquad \tilde{c} = \frac{c-1}{24}.\end{aligned} \tag{3.40}$$

We will use this expression extensively in the next section.

3.6 Computer algebra check

For $n = 1$ the expansion (3.39) reduces to $Q_1 = \Delta + n - \frac{c}{24}$, which is a simple check. A more sophisticated check is provided by Q_3, Q_5 as well as by Q_7 which are known explicitly [21] in terms of Virasoro algebra generators. Using computer algebra spectrum of Q_3 and Q_5 can be expressed in terms of boson occupation numbers n_k

$$\begin{aligned}\lambda_3 &= \tilde{\Delta}^2 + \tilde{\Delta}\left(6n - \frac{1}{2}\right) + \tilde{c}\left(\frac{\sum_k k^3 n_k}{6} + \frac{1}{1440}\right) \\ &+ \sum_k n_k^2 \left(-\frac{36\tilde{\Delta}}{\tilde{c}} - \frac{3k^2}{2} \right) + \frac{36\tilde{\Delta}(2n - \sum_k n_k)}{\tilde{c}} + \frac{1}{6} \sum_k \left(5k^3 - 9k^2\right) n_k + 3n^2 + \mathcal{O}(1/\tilde{c}),\end{aligned} \tag{3.41}$$

and $n \equiv \sum_k k\, n_k$. Similarly one can obtain an analogous expression for Q_5,

$$\lambda_5 = \tilde{\Delta}^3 + \left(15n - \frac{1}{2}\right)\tilde{\Delta}^2 + \tilde{c}\tilde{\Delta}\left(\frac{5\sum_k k^3 n_k}{6} + \frac{1}{288}\right) + \tilde{c}^2\left(\frac{\sum_k k^5 n_k}{72} - \frac{1}{36288}\right) + \tilde{c}\left(\frac{7\sum_k k^5 n_k}{36} - \frac{5}{12}\sum_k k^4 n_k^2 - \frac{5\sum_k k^4 n_k}{12} + \frac{5}{6}n\sum_k k^3 n_k + \frac{n}{288} - \frac{1}{2592}\right) - \frac{\tilde{\Delta}^2\left(180\sum_k n_k^2 + 180\sum_k n_k - 360n\right)}{\tilde{c}} + \tilde{\Delta}\left(\frac{35\sum_k k^3 n_k}{2} - \frac{35}{2}\sum_k k^2 n_k^2 - \frac{35\sum_k k^2 n_k}{2} + 25n^2 - \frac{5n}{6} + \frac{5}{72}\right) + \mathcal{O}(\tilde{c}^0)$$

These expressions are in agreement with (3.39), which serves as a non-trivial consistency check.

4 GGE partition function

Now, when we know the spectrum of quantum KdV charges explicitly we can start to reap of the benefits. We start with calculating Generalized Gibbs Ensemble (GGE) partition function which is a standard partition function of 2d CFTs decorated with KdV charges, namely,

$$Z = \mathrm{Tr}\left(e^{-\beta H - \sum_{k=2}^{\infty}\mu_{2k-1}\hat{Q}_{2k-1}}\right). \tag{4.1}$$

When all $\mu_{2k-1} = 0$ for $k \geq 2$, (4.1) is the regular partition function on a torus with the modular parameter $\tau = i\frac{\beta}{2\pi\ell}$. In particular partition function is invariant under modular transformations of τ. Turning on μ_{2k-1} would break modular invariance, although Z still exhibits some interesting properties under modular transformations [26]. In this section we focus on thermodynamic limit $\ell \to \infty$, while β, μ_{2k-1} are kept fixed. Our goal is to calculate the extensive part of free energy $F = \log Z$,

$$F = \frac{c\,\pi^2\ell}{6\,\beta} f(\beta, \mu_{2k-1}, c) + o(\ell). \tag{4.2}$$

We show how to calculate f at leading and sub-leading orders in central charge c when all fugacities are greater than or equal to zero $\mu_{2k-1} \geqslant 0$. We start with recalling how to calculate conventional partition function in thermodynamic limit. Then we discuss GGE partition function at infinite central charge. In sections 4.3 and 4.4 we move beyond infinite central charge limit and calculate sub-leading correction to the partition function. Physical implication of these results will be discussed later.

4.1 Generalized Gibbs Ensemble in the thermodynamic limit

The crucial simplification of thermodynamic limit $\ell \to \infty$ is that saddle point approximation becomes exact. We first illustrate that in the case of the conventional Gibbs ensemble

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}), \quad H = \frac{L_0 - c/24}{\ell}. \tag{4.3}$$

Here L_n is the conventional Virasoro algebra generator related to the stress tensor on the plane,

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}. \tag{4.4}$$

The sum in (4.3) goes over all states in the Hilbert space,

$$|m_i, \Delta\rangle = L_{-m_1} \dots L_{-m_k} |\Delta\rangle, \qquad m_i \geq m_j \quad \text{for} \quad i \geq j, \tag{4.5}$$

parametrized by the dimension of Virasoro primary Δ and sets $\{m_i\}$, $\sum_{i=1}^k m_i = n$, arranged in the dominance order. Using the degeneracy of H, $L_0|m_i, \Delta\rangle = (\Delta + n)|m_i, \Delta\rangle$ the partition function can be represented as a double sum,

$$Z(\beta) = \sum_\Delta \sum_{n=0} P(n)\, e^{-\frac{\beta}{\ell}(\Delta+n)}. \tag{4.6}$$

In what follows we are only interested in the extensive part of free energy, and therefore we dropped the explicitly c-dependent term in H. The sum $\sum_\Delta$ goes over all Virasoro primaries including possible multiplicities, and $P(n)$ is the number of integer partitions – Young tableaux consisting of n elements. For large n [27],

$$P(n) \approx e^{\pi\sqrt{\frac{2n}{3}}} \tag{4.7}$$

and the sum over n can be substituted by an integral. Similarly, the sum over Δ in (4.6) can be substituted by an integral multiplied by the density of primaries given by Cardy formula [28]. Combining all together gives

$$Z(\beta) = \int d\Delta\, e^{\pi\sqrt{\frac{2}{3}(c-1)\Delta}} \int dn\, e^{\pi\sqrt{\frac{2n}{3}}} e^{-\frac{\beta}{\ell}(\Delta+n)} = e^F, \quad F = \frac{c\,\pi^2\,\ell}{6\,\beta}. \tag{4.8}$$

It is easy to see that in the limit $\ell \to \infty$ the saddle point approximation is exact,

$$\mathcal{L} = \pi\sqrt{\frac{2}{3}(c-1)\Delta} + \pi\sqrt{\frac{2n}{3}} - \frac{\beta}{\ell}(\Delta + n), \tag{4.9}$$

$$\Delta^* = (c-1)\pi^2\ell^2/6\beta^2, \quad n^* = \pi^2\ell^2/6\beta^2, \qquad \left.\frac{\partial\mathcal{L}}{\partial\Delta}\right|_{\Delta^*,n^*} = \left.\frac{\partial\mathcal{L}}{\partial n}\right|_{\Delta^*,n^*} = 0, \tag{4.10}$$

$$F = \mathcal{L}(\Delta^*, n^*) = \frac{c\,\pi^2\,\ell}{6\beta}. \tag{4.11}$$

4.2 GGE at Infinite Central Charge

Our next step is to decorate the partition function by higher qKdV charges,

$$Z(\beta,\mu_{2i-1})=\mathrm{Tr}(e^{-\beta H-\sum_k \mu_{2k-1}\hat{Q}_{2k-1}}). \tag{4.12}$$

We start our analysis with $\hat{Q}_3$. The expression for $\hat{Q}_3$ in terms of Virasoro generators can be found in [18] (in our case expression for $\hat{Q}_3$ is different by an overall factor $1/\ell^3$). Using the explicit form of $\hat{Q}_3$ we split it into two parts as follows

$$\hat{Q}_3=Q_3^l+\tilde{Q}_3 \tag{4.13}$$

$$\ell^3 Q_3^l=\left(L_0^2-\frac{c+2}{12}L_0+\frac{c(5c+22)}{2880}\right),\quad \ell^3\tilde{Q}_3=2\sum_{n=1}^{\infty}L_{-n}L_n. \tag{4.14}$$

The term $\tilde{Q}_3$ is defined such that it annihilates primary states, $\tilde{Q}_3|\Delta\rangle=0$, while the rest is degenerate and depends only on the combination $\Delta+n$. Using scaling with ℓ of the saddle point values $\Delta^*\sim\ell^2$ and $n^*\sim\ell^2$ we immediately find that L_0-independent term from Q_3^l would give $1/\ell^3$ contribution to free energy, while $-\frac{c+2}{12}L_0$ will contribute as $\sim 1/\ell$. Hence in the thermodynamic limit these terms can be neglected. Assuming for simplicity that only μ_3 is turned on we get,

$$Z(\beta,\mu_3)=\sum_{\Delta}\sum_{n=0}P(n)\,e^{-\frac{\beta}{\ell}(\Delta+n)-\frac{\mu_3}{\ell^3}(\Delta+n)^2+\tilde{\mathcal{L}}(c,\mu_3/\ell^3,\Delta,n)}, \tag{4.15}$$

$$e^{\tilde{\mathcal{L}}(c,\mu_3/\ell^3,\Delta,n)}\equiv\frac{1}{P(n)}\sum_{\{m\}=n}\langle m_i,\Delta|e^{-\mu_3\tilde{Q}_3}|m_i,\Delta\rangle. \tag{4.16}$$

In (4.16) the sum is over sets $\{m_i\}$ satisfying $\sum_i m_i=n$, i.e. over the partitions of n.

A crucial simplification, which will be justified in the next section, is that $\tilde{Q}_3$ does not contribute to free energy at leading order in $1/c$ expansion. Hence at infinite central charge one can simply take $\tilde{\mathcal{L}}$ to be zero,

$$Z(\beta,\mu_3)=\int d\Delta\int dn\,e^{\mathcal{L}}, \tag{4.17}$$

$$\mathcal{L}=\pi\sqrt{\frac{2}{3}(c-1)\Delta}+\pi\sqrt{\frac{2n}{3}}-\frac{\beta}{\ell}(\Delta+n)-\frac{\mu_3}{\ell^3}(\Delta+n)^2. \tag{4.18}$$

Adding higher charges can be done in a similar way. Thus Q_5, also given in [18], can be split into

$$Q_5=Q_5^l+\tilde{Q}_5 \tag{4.19}$$

$$\ell^5 Q_5^l=\left(L_0^3-\frac{c+4}{8}L_0^2+\frac{(c+2)(3c+20)}{576}L_0-\frac{c(3c+14)(7c+68)}{290304}\right), \tag{4.20}$$

$$\ell^5\tilde{Q}_5=\sum_{\substack{n_1+n_2+n_3=0\\ n_1\text{ or }n_2\text{ or }n_3\neq 0}}:L_{n_1}L_{n_2}L_{n_3}:+\sum_{n=1}^{\infty}\left(\frac{c+11}{6}n^2-1-\frac{c}{4}\right)L_{-n}L_n+\frac{3}{2}\sum_{r=1}^{+\infty}L_{1-2r}L_{2r-1}.$$

Normal ordering in $:L_{n_1}L_{n_2}L_{n_3}:$ means that the operators ordered to satisfy $n_3 \geq n_2 \geq n_1$. Again $\tilde{Q}_5$ is chosen to annihilate primary states, while Q_5^l is a function of L_0. It is easy to see from ℓ-scaling of saddle point values of Δ, n that only L_0^3 term from Q_5^l contributes to the extensive part of free energy. Similarly to the case of Q_3, we drop $\tilde{Q}_5$ at leading order in c. The rest is straightforward and can be generalized to all Q_{2k-1}. After integrating over n the partition function can be reduced to the integral over $E = \Delta + n$,

$$Z(\beta, \mu_{2k-1}) = \int dE\, e^{\mathcal{L}_E}, \quad \mathcal{L}_E = \pi\sqrt{\frac{2}{3}cE} - \frac{\beta}{\ell}E - \frac{\mu_3}{\ell^3}E^2 - \frac{\mu_5}{\ell^5}E^3 - \dots. \tag{4.21}$$

It is easy to see than that free energy will depend on the inverse temperature β and fugacities μ_i only through the combinations

$$F = \frac{c\,\pi^2\ell}{6\,\beta} f_0(t_{2k-1}), \qquad t_{2k-1} = \left(\frac{c\,\pi^2}{6\,\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}. \tag{4.22}$$

Free energy F admits perturbative expansion in t_{2i-1} at any value of c, while expansion in μ_{2k-1} breaks down for large central charge. In principle f_0 is given by the algebraic equation specifying the saddle point E^* of (4.21),

$$\mathcal{L}_E = \frac{c\,\pi^2\ell}{6\,\beta} s_0, \qquad E^* = \frac{c\,\pi^2\ell}{6\,\beta} e^*, \tag{4.23}$$

$$s_0 = 2\sqrt{e} - e - t_3 e^2 - t_5 e^3 - \dots, \tag{4.24}$$

$$f_0 = s_0(e^*), \qquad \left.\frac{\partial s_0}{\partial e}\right|_{e^*} = 0. \tag{4.25}$$

In practice perturbative expansion of f_0 is easier to recover iteratively. We notice that at leading order in c the expectation values of Q_{2k-1} are given by $\ell(E^*/\ell^2)^k$. In terms of the partition function (4.21) this can be rewritten as a differential equality

$$-\ell^{-1}\partial_{\mu_{2k-1}} \log Z = (-\ell^{-1}\partial_\beta \log Z)^k, \tag{4.26}$$

or in terms of variables t_{2k-1},

$$(f_0 + 3t_3\partial_{t_3} f_0 + 5t_5\partial_{t_5} f_0 + \dots)^2 + \partial_{t_3} f_0 = 0\ , \tag{4.27}$$

$$(f_0 + 3t_3\partial_{t_3} f_0 + 5t_5\partial_{t_5} f_0 + \dots)^3 + \partial_{t_5} f_0 = 0\ , \tag{4.28}$$

$$\dots$$

For the Taylor expansion of f_0 equations (4.27),(4.28), etc. yield iterative relations which can be easily solved,

$$f_0 = 1 - t_3 + 4t_3^2 - 24t_3^3 - t_5 + 12t_3t_5 - 132t_3^2t_5 + 9t_5^2 - 234t_3t_5^2 - 135t_5^3 + \dots \tag{4.29}$$

This result can be generalized in order to include all higher qKdV charges. The function f_0 can be elegantly written in terms of all t_{2k-1} as follows:

$$f_0 = 1 + \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n=2}^{\infty} 2\frac{(-1)^n}{n!} \frac{(2K-n)!}{(2K-2n+2)!} \prod_{i=1}^{n} k_i\, t_{2k_i-1}, \quad K \equiv \sum_i k_i. \tag{4.30}$$

This expression was originally found in [29].

4.3 GGE at Finite Central Charge

When the central charge is finite, extensive part of free energy acquires $1/c$ corrections,

$$F = \frac{c\,\pi^2 \ell}{6\,\beta} \left(f_0 + \frac{f_1}{c} + \frac{f_2}{c^2} + \ldots \right). \tag{4.31}$$

Functions f_k admit Taylor expansion in terms of t_{2k-1} (4.22). For $k \geq 2$, f_k depends only on t_{2i-1}, $i \geq k$, and start with a term linear in t_{2k-1}. Thus, if expressed in terms of μ_{2k-1}, extensive part of free energy is polynomial in c. To calculate $f_1, f_2, \ldots$ we need to take into account additional contribution to the effective action,

$$e^{\tilde{\mathcal{L}}(c,\mu_{2k-1}/\ell^{2k-1},\Delta,n)} \equiv \frac{1}{P(n)} \sum_{\{m\}=n} \langle m_i, \Delta | e^{-\mu_3 \tilde{Q}_3 - \mu_5 \tilde{Q}_5 - \cdots} | m_i, \Delta \rangle. \tag{4.32}$$

4.3.1 $1/c$ corrections from $\tilde{Q}_3$

We start with the case when only $\mu_3 \neq 0$, (4.16). It is easy to see that the operator $\tilde{Q}_3$ written in the basis (4.5) is not more than linear in c and Δ,

$$\tilde{Q}_3 = c\,\tilde{Q}_3^c + \Delta\,\tilde{Q}_3^\Delta + \tilde{Q}_3^{(0)}. \tag{4.33}$$

The key observation is that matrices $\tilde{Q}_3^c, \tilde{Q}_3^\Delta$ are lower-triangular. Its diagonal elements are

$$\ell^3 \tilde{Q}_3^c |m_i, \Delta\rangle = \lambda |m_i, \Delta\rangle + \ldots, \qquad \lambda = \frac{1}{6}\left(\sum_i m_i^3 - m_i \right), \tag{4.34}$$

$$\ell^3 \tilde{Q}_3^\Delta |m_i, \Delta\rangle = \nu |m_i, \Delta\rangle + \ldots, \qquad \nu = 4\,n. \tag{4.35}$$

Because of the triangular form, $c\,\lambda + \Delta\,\nu$ are the eigenvalues of $c\,\tilde{Q}_3^c + \Delta\,\tilde{Q}_3^\Delta$. Moreover, these diagonal elements are exactly the eigenvalues λ_3 (3.40) of $\hat{Q}_3$ at sub-leading order. This is the reason why we can parametrize qKdV spectrum at large c with Young tableau (3.26),

$$\lambda_3 = \left(\Delta - \frac{c}{24}\right)^2 + \left(\Delta - \frac{c}{24}\right)\left[6\sum k n_k - \frac{1}{6}\right] + c\left[\frac{1}{6}\sum k^3 n_k + + \frac{1}{1440}\right]. \tag{4.36}$$

Here the sum goes over n_k rather than m_i This sum can be understood as a rewriting in the free boson representation. Namely, each set m_i will be parametrized by the set of integer numbers n_k, where n_k is the number of times natural number k appears in the set m_i. Then the sum over all sets $\{m_i\}$ is equivalent to the sum over all n_k and

$$\sum_i m_i = \sum_{k=1}^{\infty} k\, n_k, \qquad \sum_i m_i^3 = \sum_{k=1}^{\infty} k^3 n_k. \tag{4.37}$$

Further, we will use both representation.

To estimate contribution of each term in (4.33) toward free energy in the section 4.3.3 we calculate

$$\ell^3 \langle \tilde{Q}_3 \rangle_{\Delta,n} \equiv \frac{\ell^3}{P(n)} \sum_{\{m\}=n} \langle m_i, \Delta | \tilde{Q}_3 | m_i, \Delta \rangle = (a_0 c + b_0) n^2 + 4\Delta n + O(\ell^3), \tag{4.38}$$

in the limit of infinite n, assuming $n, \Delta \sim \ell^2$. The n-independent coefficients

$$a_0 = \frac{2}{5}, \qquad b_0 = 4. \tag{4.39}$$

We note that leading n-scaling in (4.38) is fixed to be n^2 lest the contribution of $\tilde{Q}_3$ diverge in the thermodynamic limit $\ell \to \infty$. At leading order in c this is easy to see directly from the sum over Young tableaux in (4.38),

$$\frac{1}{P(n)} \sum_{\{m\}=n} \left(\frac{1}{6} \sum_i m_i^3 \right) = a_0 n^2 + O(n). \tag{4.40}$$

For large n there are exponentially many ways to represent n as a sum of integers with the typical partition consisting of $\sim \sqrt{n}$ terms with each term being of order $m_i \sim \sqrt{n}$. From here it immediately follows that $\sum_i m_i^3 \sim \sqrt{n} \times n^{3/2} \sim n^2$. After substituting scaling of saddle-point values $\Delta^* \sim c\ell^2$, $n^* \sim \ell^2$ into (4.38) we find that both $c\,\tilde{Q}_3^c$ and $\Delta\, \tilde{Q}_3^{\Delta}$ contribute toward f_1 and potentially to f_2, while $\tilde{Q}_3^{(0)}$ contributes to f_2 only. So far we are interested only in f_1 we can simplify (4.16) to be

$$e^{\tilde{\mathcal{L}}(c,\, \mu_3/\ell^3, \Delta, n)} \equiv \frac{e^{-\frac{\mu_3}{\ell^3} 4n\Delta}}{P(n)} \sum_{\{m\}=n} e^{-\frac{\mu_3}{\ell^3} \frac{c}{6} \sum_i m_i^3}, \tag{4.41}$$

$$\tilde{\mathcal{L}} = -\mu_3/\ell^3\, 4\, n\, \Delta - a_0\, (c\, \mu_3/\ell^3) n^2 - a_1\, (c\, \mu_3/\ell^3)^2 n^{7/2} - a_2\, (c\, \mu_3/\ell^3)^3 n^5 + \ldots \tag{4.42}$$

The expansion (4.42) assumes $\ell^3/(c\,\mu_3) \gg n \gg 1$. Sum over Young tableaux (4.41) provides a non-perturbative definition of $\tilde{\mathcal{L}}$. It is a simplified version of the sums appearing in [30].

In principle effective action (4.42) together with (4.18) completely determines $f_1(t_3)$ via maximization over Δ and n, but there is a direct way to obtain f_1 bypassing this step. We first notice that at leading order in c effective action (4.18) as a function of Δ reduces to (4.21) and therefore saddle-point value of Δ^* is fixed by (4.24) independently of the value of n^*,

$$\Delta^* = \frac{c\pi^2\ell^2}{6\,\beta^2}\left(e^* + O(1/c)\right). \tag{4.43}$$

In case when only t_3 is "turned on" e^* can be found explicitly,

$$\begin{aligned} e^* &= \frac{(t_3 - (t_3^3 + 27t_3^4 + 3^{3/2}t_3^{7/2}\sqrt{2+27t_3})^{1/3})^2}{6t_3^2(t_3^3 + 27t_3^4 + 3^{3/2}t_3^{7/2}\sqrt{2+27t_3})^{1/3}} \\ &= 1 - 4t_3 + 28t_3^2 - 240t_3^3 + 2288t_3^4 - 23296t_3^5 + O\left(t_3^6\right). \end{aligned}$$

Then leading $1/c$ correction to free energy can be found now by plugging Δ^* back into (4.18) and keeping $O(c^0)$ terms,

$$e^{\frac{\pi^2\ell}{6\beta}f_1} = e^{-\frac{\pi}{2}\sqrt{\frac{\Delta^*}{6c}}} \sum_{\{m_i\}} e^{-\frac{\beta}{\ell}n - 6\frac{\mu_3}{\ell^3}n\Delta^* - \frac{\mu_3}{\ell^3}\frac{c}{6}\sum_i m_i^3}. \tag{4.44}$$

Here $n = \sum_i m_i$ and the sum goes over all Young tableaux. This sum can be calculated by rewriting (4.44) using free boson representation (4.37). For any coefficients $x, y > 0$ we find

$$\sum_{\{m_i\}} e^{-x\sum_i m_i - y\sum_i m_i^3} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\ldots e^{-\sum_k n_k(xk+yk^3)} = \prod_{k=1}^{\infty}\left(1 - e^{-xk-yk^3}\right)^{-1}. \tag{4.45}$$

This infinite product can be consequently rewritten as an exponent of the sum of logarithms. Going back to (4.44) this gives

$$\frac{\pi^2\ell}{6\beta}f_1 = -\frac{\pi^2\ell}{6\beta}\sqrt{e^*} - \sum_{k=1}^{\infty}\log\left(1 - e^{-\frac{\beta}{\ell}(1+6t_3e^*)k - \frac{t_3}{\pi^2}\left(\frac{\beta}{\ell}\right)^3k^3}\right). \tag{4.46}$$

The effective variables in (4.46) is the combination $\beta k/\ell$ and in the limit $\ell \to \infty$ the sum over k can be substituted by an integral, yielding

$$f_1 = -\sqrt{e^*} - \frac{6}{\pi^2}\int_0^{\infty} dk \log\left(1 - e^{-(1+6t_3e^*)k - t_3k^3/\pi^2}\right). \tag{4.47}$$

A few first terms in Taylor series expansion of f_1 can be easily calculated by expanding this expression in t_3,

$$f_1 = -\frac{22}{5}t_3 + \frac{2096}{35}t_3^2 - \frac{4464}{5}t_3^3 + \frac{82304}{5}t_3^4 + O\left(t_3^5\right). \tag{4.48}$$

This agrees with the perturbative calculation of [31].

4.3.2 $1/c$ corrections from $\tilde{Q}_5$

Adding μ_5 to consideration is straightforward. We notice that $\tilde{Q}_5$ is not more than quadratic in c and Δ,

$$\tilde{Q}_5 = c^2\tilde{Q}_5^{cc} + c\,\Delta\tilde{Q}_5^{c\Delta} + \Delta^2\tilde{Q}_5^{\Delta\Delta} + c\,\tilde{Q}_5^{c} + \Delta\tilde{Q}_5^{\Delta} + \tilde{Q}_5^{(0)}, \tag{4.49}$$

and all three matrices $\tilde{Q}^{cc}$, $\tilde{Q}^{c\Delta}$, $\tilde{Q}_5^{\Delta\Delta}$ are lower-triangular in the basis (4.5). Their diagonal matrix elements are easy to calculate,

$$\ell^5\tilde{Q}_5^{cc}|m_i,\Delta\rangle = \alpha|m_i,\Delta\rangle + \ldots, \quad \alpha = \frac{1}{12}\left(\sum_i \frac{m_i^5}{6} - \frac{5m_i^3}{12} + \frac{m_i}{4}\right), \tag{4.50}$$

$$\ell^5\tilde{Q}_5^{c\Delta}|m_i,\Delta\rangle = \delta|m_i,\Delta\rangle + \ldots, \qquad \delta = \sum_i \frac{5}{6}m_i^3 - m_i, \tag{4.51}$$

$$\ell^5\tilde{Q}_5^{\Delta\Delta}|m_i,\Delta\rangle = \gamma|m_i,\Delta\rangle + \ldots, \qquad \gamma = 12\,n. \tag{4.52}$$

Only these three matrices contribute toward f_1 via

$$e^{\tilde{\mathcal{L}}(c,\,\mu_3/\ell^3,\,\mu_5/\ell^5,\Delta,n)} \equiv \frac{e^{-\frac{\mu_3}{\ell^3}4n\Delta - \frac{\mu_5}{\ell^5}12n\Delta^2}}{P(n)} \sum_{\{m\}=n} e^{-\left(\frac{c\mu_3}{6\,\ell^3}+\frac{5c\Delta\mu_5}{6\,\ell^5}\right)\sum_i m_i^3 - \frac{c^2\mu_5}{72\ell^5}\sum_i m_i^5}. \tag{4.53}$$

Next steps are completely analogous to the case with only $\hat{Q}_3$. At leading order in c saddle point value of Δ is fixed by

$$s_0 = 2\sqrt{e} - e - t_3e^2 - t_5e^3, \tag{4.54}$$

$$f_0 = s_0(e^*), \qquad \left.\frac{ds_0}{de}\right|_{e^*} = 0, \qquad \Delta^* = \frac{c\pi^2\ell^2}{6\,\beta^2}e^*, \tag{4.55}$$

while f_1 is given by

$$f_1 = -\sqrt{e^*} - \frac{6}{\pi^2}\int_0^\infty dk \log\left(1 - e^{-(1+6t_3e^*+15t_5(e^*)^2)k - (t_3+5t_5e^*)k^3/\pi^2 - \frac{1}{2}t_5k^5/\pi^4}\right). \tag{4.56}$$

A few first terms in Taylor series expansion are

$$f_1 = -\frac{22}{5}t_3 - \frac{302}{21}t_5 + \frac{2096}{35}t_3^2 + \frac{14328}{35}t_3t_5 + \frac{51168}{77}t_5^2 + \ldots \tag{4.57}$$

Generalization to include higher charges is conceptually straightforward. We will give the full expression for f_1 in terms of all t_{2k-1} in section 4.4.

4.3.3 Computation of a_0, b_0

In this section we show how to calculate

$$\ell^3\langle\tilde{Q}_3\rangle_{\Delta,n} \equiv \frac{\ell^3}{P(n)}\sum_{\{m\}=n}\langle m_i,\Delta|\tilde{Q}_3|m_i,\Delta\rangle = (a_0 c + b_0)n^2 + 4\Delta n + O(\ell^3), \quad (4.58)$$

in the limit when $\Delta, n \sim \ell^2$. An analogous calculation can be also found in [26]. Recall that $\tilde{Q}_3 = 2\sum L_{-k}L_k$. The expectation value $\langle\tilde{Q}_3\rangle_{\Delta,n}$ can be thought of as trace $q^{-n-\Delta}\,\mathrm{Tr}\Big(q^{L_0}\tilde{Q}_3\Big)$ over a subspace with fixed n and Δ spanned by the states (4.5). Using cyclic property of trace one can easily get [32]

$$\begin{aligned}\langle L_{-k}L_k\rangle_{\Delta,n} = q^k\langle L_k L_{-k}\rangle_{\Delta,n} = \frac{q^k}{1-q^k}\langle[L_k,L_{-k}]\rangle_{\Delta,n} = \\ = \frac{q^k}{1-q^k}\left(2k\langle L_0\rangle_{\Delta,n} + \frac{c}{12}(k^3-k)\right),\end{aligned} \quad (4.59)$$

where $q = e^{-\frac{\beta}{\ell}}$. Summing (4.59) over k one obtains

$$\langle\tilde{Q}_3\rangle_{\Delta,n} = 4\sigma_1\langle L_0\rangle_{\Delta,n} + \frac{c}{6}(\sigma_3-\sigma_1), \quad (4.60)$$

where

$$\sigma_1 = \sum_k \frac{kq^k}{1-q^k}, \quad (4.61)$$

$$\sigma_3 = \sum_k \frac{k^3q^k}{1-q^k}. \quad (4.62)$$

Expectation value $\langle L_0\rangle_{\Delta,n}$ is just equal to $n+\Delta$ and the sums σ_1, σ_3 can be evaluated in thermodynamic limit by replacing the sum over k by an integral $\sum_k \to \int dk$. In this limit

$$\sigma_1 \to \frac{\ell^2\pi^2}{6\beta^2}, \quad (4.63)$$

$$\sigma_3 \to \frac{\ell^4\pi^4}{15\beta^4}. \quad (4.64)$$

The final step is to recall that if we sum $\mathrm{Tr}\Big(q^{L_0}\tilde{Q}_3\Big)$ over n main contribution will be given by a particular saddle point value $n = \frac{\ell^2\pi^2}{6\beta^2}$. Thus substituting $\frac{\ell^2\pi^2}{6\beta^2}$ by n we get

$$\langle\tilde{Q}_3\rangle_{\Delta,n} = 4n(n+\Delta) + \frac{2}{5}c\,n^2. \quad (4.65)$$

As a result, $a_0 = \frac{2}{5}$ and $b_0 = 4$. It is straightforward but tedious to generalize the above calculation to higher orders to obtain a_1, a_2 etc. from (4.42).

4.3.4 Effective action at $1/c$ decorated with $\hat{Q}_3$

In this section we show how to calculate the effective action (4.42), which is a partition function decorated by the first non-trivial qKdV charge Q_3 restricted to a particular large descendant level n,

$$e^{\tilde{\mathcal{L}}(c,\,\mu_3/\ell^3,\Delta,n)} \equiv \frac{e^{-\frac{\mu_3}{\ell^3}4n\Delta}}{P(n)} \sum_{\{m\}=n} e^{-\frac{\mu_3}{\ell^3}\frac{c}{6}\sum_i m_i^3}. \tag{4.66}$$

We remind the reader that first $1/c$ correction to free energy f_1 calculated in section 4.3.1 is given by (compare with (4.44))

$$e^{\frac{\pi^2\ell}{6\beta}f_1} = e^{-\frac{\pi}{2}\sqrt{\frac{\Delta^*}{6c}}} \sum_n P(n)\, e^{-\frac{\beta}{\ell}n-\frac{\mu_3}{\ell^3}2n\Delta^*+\tilde{\mathcal{L}}(c,\,\mu_3/\ell^3,\Delta^*,n)}. \tag{4.67}$$

To calculate $\tilde{\mathcal{L}}$ we introduce an auxiliary partition function of the same kind, which depends on arbitrary parameter x,

$$e^{\mathrm{F}(x)} = \sum_n P(n)\, e^{-\frac{x}{\ell}n+\frac{\mu_3}{\ell^3}4n\Delta+\tilde{\mathcal{L}}(c,\,\mu_3/\ell^3,\Delta,n)}. \tag{4.68}$$

In the limit of large ℓ this sum is saturated at some saddle point n^*,

$$\mathrm{F}(x) = \pi\sqrt{\frac{2n^*}{3}} - \frac{x}{\ell}n^* + \frac{\mu_3}{\ell^3}4n^*\Delta + \tilde{\mathcal{L}}(c,\,\mu_3/\ell^3,\Delta,n^*), \tag{4.69}$$

which is related to x and μ_3 via $n^* = -\ell\frac{dF}{dx}$. "Free energy" F can be calculated exactly the same way f_1 was calculated in section 4.3.1,

$$\mathrm{F}(x) = -\ell\int_0^\infty dk \ln\left(1-e^{-xk-\frac{c}{6}\mu_3 k^3}\right). \tag{4.70}$$

After changing the integration variable $k \to k/x$ and expanding in μ_3 one finds

$$\mathrm{F}(x) = \frac{\ell}{x}\left(\frac{\pi^2}{6} - \frac{\pi^4 c\mu_3}{90x^3} + \frac{2\pi^6c^2\mu_3^2}{189x^6} - \frac{4\pi^8c^3\mu_3^3}{135x^9} + \frac{40\pi^{10}c^4\mu_3^4}{243x^{12}} + \dots\right) \tag{4.71}$$

Using the explicit expression (4.70) we can relate n^* and x,

$$\frac{n^*}{\ell^2} = \frac{d}{dx}\int_0^\infty dk \ln\left(1-e^{-xk-\frac{c}{6}\mu_3 k^3}\right), \tag{4.72}$$

and solve this equation perturbatively with respect to x by expanding in μ_3,

$$x = \frac{\pi\ell}{\sqrt{6n^*}} - \frac{4c\mu_3 n^*}{5\ell^2} + \frac{144\sqrt{6}c^2\mu_3^2(n^*)^{5/2}}{25\pi\ell^5} - \frac{20736c^3\mu_3^3(n^*)^4}{25\pi^2\ell^8} + \frac{4762368\sqrt{6}c^4\mu_3^4(n^*)^{11/2}}{125\pi^3\ell^{11}} + \dots$$

Plugging this back into (4.69) and finally renaming n^* into n gives (compare with (4.42))

$$\frac{\mu_3}{\ell^3}4n\Delta + \tilde{\mathcal{L}}(c,\, \mu_3/\ell^3, \Delta, n) = -\pi\sqrt{\frac{2n}{3}} + \frac{x(n)}{\ell}n + \mathrm{F}(x(n)) \tag{4.73}$$

$$= -\frac{2}{5}\,(c\,\mu_3/\ell^3)n^2 + \frac{288\sqrt{6}}{175\pi}\,(c\,\mu_3/\ell^3)^2 n^{7/2} - \tag{4.74}$$

$$-\frac{20736}{125\pi^2}\,(c\,\mu_3/\ell^3)^3 n^5 \,+\, \frac{732672\sqrt{6}}{125\pi^3}\,(c\,\mu_3/\ell^3)^4 n^{13/2}\ldots$$

Generalization to include $\hat{Q}_5$ and higher charges would be tedious but straightforward.

4.4 Full generalized partition function

In this section we extend the calculation of section 4.3 to all qKdV charges $\hat{Q}_{2k-1}$. Conceptually, this is the same calculation. We sum over "free bosons" using the leading part of qKdV spectrum (3.40).

Given that all $\hat{Q}_{2k-1}$ mutually commute, the generalized partition function (4.1) is given by the sum over primaries Δ and sets (Young tables) $\{m_i\}$, parametrizing descendants via (4.5),

$$Z = \sum_{\Delta}\sum_{\{m_i\}} \exp\left(-\sum_{k=1}^{\infty}\frac{\mu_{2k-1}}{\ell^{2k-1}}\left(\Delta^k + \sum_{p=0}^{k-1}\sum_i m_i^{2p+1} c^p \Delta^{k-1-p}\,\tilde{\xi}_k^p + O(c^{k-2})\right)\right) \tag{4.75}$$

where

$$\tilde{\xi}_k^p = 24^{-p}\frac{(2k-1)\Gamma(k+1)\Gamma(1/2)}{2\Gamma(p+3/2)\Gamma(k-p)}. \tag{4.76}$$

Note, that we neglected irrelevant in thermodynamic limit terms. At large central charge sum over Δ can be substituted by an integral

$$\sum_{\Delta} \to \int d\Delta\, e^{\pi\sqrt{2c'\Delta/3}}, \qquad c' \equiv c - 1, \tag{4.77}$$

where the density of primaries follows from Cardy formula [28, 33]. It is convenient to introduce σ via

$$\Delta = \frac{c'\pi^2\ell^2}{6\,\beta^2}\sigma. \tag{4.78}$$

In this section we will do an expansion in $1/c'$. Since at leading order $c = c' + O(1)$, the structure of λ remains the same: Δ^k contributes as $(c')^k$ while $c^p\Delta^{k-1-p}$ terms

contribute as $(c')^{k-1}$. Going from the sets $\{m_i\}$ to free boson representation (4.37), the partition function reduces to that one of non-interacting auxiliary bosons

$$Z(\beta,t)=\int d\sigma\exp\left\{\frac{c'\pi^2\ell}{6\beta}\left(2\sqrt{\sigma}-\sum_{k=1}^{\infty}t_{2k-1}\sigma^k\right)\right\}\sum_{n_1,n_2,\ldots}e^{-\sum_{r=1}^{\infty}n_rM_r+O(1/c')} \tag{4.79}$$

$$\log Z\equiv F=\frac{\pi^2\ell}{6\beta}\left(c'f_0(t)+f_1(t)+O(1/c')\right), \tag{4.80}$$

$$t_{2k-1}=\left(\frac{\pi^2c'}{6\beta^2}\right)^{k-1}\frac{\mu_{2k-1}}{\beta},\qquad t_1\equiv 1, \tag{4.81}$$

where the spectrum of bosons is given by

$$M_r=\sum_{k=1}^{\infty}t_{2k-1}\sigma^{k-1}\sum_{p=0}^{k-1}\xi_k^p\left(\frac{6}{\pi^2\sigma}\right)^p\left(\frac{\beta r}{\ell}\right)^{2p+1}= \tag{4.82}$$

$$\frac{\beta r}{\ell}\sum_{k=1}^{\infty}t_{2k-1}\sigma^{k-1}k(2k-1)\,{}_2F_1\left(1,1-k,3/2,-\frac{1}{\sigma}\left(\frac{\beta r}{2\pi\ell}\right)^2\right). \tag{4.83}$$

In (7.40) we write the partition function as a function of β, t_{2k-1}. For the given fixed β, t_{2k-1} the terms contributing as $(c')^{k-2}$ to eigenvalues of Q_{2k-1} contribute to free energy as $1/c'$. Our scope is to calculate free energy up to the first two orders in $1/c'$ expansion, i.e. only keep the terms which survive in the $c'\to\infty$ limit. Hence $O(1/c')$ terms can be neglected.

Up to $1/c'$ corrections the value of σ is determined via saddle point approximation of

$$Z_0(\beta,t)=\exp\left\{\frac{c'\pi^2\ell}{6\beta}f_0\right\}=\int d\sigma\exp\left\{\frac{c'\pi^2\ell}{6\beta}\left(2\sqrt{\sigma}-\sum_{k=1}^{\infty}t_{2k-1}\sigma^k\right)\right\}, \tag{4.84}$$

while the remaining sum over the boson occupation numbers n_r in (7.40) "takes" saddle point value of σ as an input. The saddle point equation

$$\sqrt{\sigma}=\sum_{k=1}^{\infty}t_{2k-1}\sigma^k k, \tag{4.85}$$

can be solved in terms of an infinite series

$$\sigma=1+\sum_{n=1}^{\infty}\sum_{k_1,\ldots,k_n=2}^{\infty}2\frac{(-1)^n}{n!}\frac{(2K-n+1)!}{(2K-2n+2)!}\prod_{i=1}^{n}k_i\,t_{2k_i-1},\qquad K\equiv\sum_i k_i, \tag{4.86}$$

yielding (expansion (4.88) was found in [29]),

$$f_0 = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k(2k-1), \tag{4.87}$$

$$f_0 = 1 + \sum_{n=1}^{\infty}\sum_{k_1,\ldots,k_n=2}^{\infty} 2\frac{(-1)^n}{n!}\frac{(2K-n)!}{(2K-2n+2)!}\prod_{i=1}^{n} k_i\, t_{2k_i-1}, \quad K \equiv \sum_i k_i. \tag{4.88}$$

With σ being fixed, the remaining part of the partition function describes some auxiliary non-interacting bosons

$$\frac{\pi^2\ell}{6\beta} f_1 = \log \sum_{n_1,n_2,\ldots} e^{-\sum_{r=1}^{\infty} n_r M_r} = -\sum_{r=1}^{\infty} \log\left(1 - e^{-M_r}\right). \tag{4.89}$$

In the thermodynamic limit $\ell \to \infty$ summation over r can be substituted by integration (Thomas–Fermi approximation), yielding (4.90),

$$Z = e^F, \qquad F = \frac{\pi^2\ell}{6\,\beta}\left(c' f_0 + f_1 + O(1/c')\right), \tag{4.90}$$

$$f_0 = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k(2k-1), \qquad \sqrt{\sigma} = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k k,$$

$$f_1 = -\frac{12}{\pi}\int_0^{\infty} d\kappa \log\left(1 - e^{-2\pi\kappa\gamma}\right), \quad \gamma = \sum_{k=1}^{\infty} t_{2k-1}\sigma^{k-1}k(2k-1)\,{}_2F_1(1, 1-k, 3/2, -\kappa^2/\sigma),$$

$$c' = c - 1, \qquad t_{2k-1} = \left(\frac{\pi^2 c'}{6\beta^2}\right)^{k-1}\frac{\mu_{2k-1}}{\beta}, \qquad t_1 \equiv 1.$$

Here $\sigma(t_1, t_3, \ldots)$ is a function which satisfies $\sqrt{\sigma} = \sum_{k=1}^{\infty} t_{2k-1}\sigma^k k$. It can be expressed explicitly in terms of an infinite power series in t_{2k-1}, see (4.86).

4.4.1 Alternative representation of the partition function

The answer (4.90) was derived assuming $\beta \neq 0$ and $\mu_1 = \beta$ enters the expression for free energy differently from all other chemical potentials. In this section we obtain the answer for free energy F in another "coordinate patch," assuming some other chemical potential μ_{2r-1} for a given r is non-zero, $\mu_{2r-1} \neq 0$, while the rest of chemical potentials, including $\mu_1 = \beta$, could be zero.

Let us introduce $c'^{r-1}\mu_{2r-1} = \lambda \neq 0$ and the following set of independent variables

$$\tau_{2k-1} = \frac{\mu_{2k-1}}{\mu_{2r-1}} c'^{k-r}\left(\frac{\pi^2}{6\lambda^2 r^2}\right)^{\frac{k-r}{2r-1}}, \quad \tau_{2r-1} \equiv 1, \tag{4.91}$$

and functions $f_i(\tau)$, $\sigma(\tau)$,

$$F = c'\ell\lambda \left(\frac{\pi^2}{6\lambda^2 r^2}\right)^{\frac{r}{2r-1}} (f_0 + f_1/c' + O(1/c'^2)), \quad \Delta = c'\ell^2 \left(\frac{\pi^2}{6\lambda^2 r^2}\right)^{\frac{1}{2r-1}} \sigma. \quad (4.92)$$

Using these notations the expression for f_0 is as follows

$$f_0 = 2r\sqrt{\sigma} - \sum_{k=1}^{\infty} \tau_{2k-1}\sigma^k = \sum_{k=1} (2k-1)\tau_{2k-1}\sigma^k, \quad (4.93)$$

where the last equality holds "on-shell,"

$$r\sigma^{1/2} = \sum_{k=1}^{\infty} \tau_{2k-1} k\, \sigma^k, \qquad \sigma = 1 - \frac{2}{r(2r-1)} \sum_{k\neq r} k\, \tau_{2k-1} + \dots \quad (4.94)$$

Finally, the expression for f_1,

$$f_1 = -\frac{12r}{\pi} \int_0^{\infty} d\kappa \log\left(1 - \exp\left\{-\frac{2\pi}{r}\kappa\gamma\right\}\right), \quad (4.95)$$

$$\gamma = \sum_{k=1} \tau_{2k-1} k(2k-1)\sigma^{k-1}\, {}_2F_1(1, 1-k, 3/2, -\kappa^2/\sigma). \quad (4.96)$$

4.4.2 $1/c$ versus $1/c'$ expansion

In section 4.3.4 we were discussing free energy in $1/c$ expansion

$$F = \frac{\pi^2\ell}{6\beta}\left(c\tilde{f}_0(\tilde{t}) + \tilde{f}_1(\tilde{t}) + O(1/c)\right), \quad (4.97)$$

using variables

$$\tilde{t}_{2k-1} = \left(\frac{\pi^2 c}{6\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}. \quad (4.98)$$

But here we used on $1/c'$ expansion

$$F = \frac{\pi^2\ell}{6\beta}\left(c' f_0(t) + f_1(t) + O(1/c')\right), \quad (4.99)$$

and the variables

$$t_{2k-1} = \left(\frac{\pi^2 c'}{6\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}. \quad (4.100)$$

Here we outline the relation between these two expansion schemes. Using

$$t_{2k-1} = \tilde{t}_{2k-1}\left(1 - \frac{1}{c}\right)^{k-1} \tag{4.101}$$

we readily find

$$\tilde{f}_0(t) = f_0(t), \tag{4.102}$$

and

$$\tilde{f}_1(t) = -f_0(t) - \sum_{k=1}^{\infty}(k-1)t_{2k-1}\frac{\partial f_0(t)}{\partial t_{2k-1}} + f_1(t). \tag{4.103}$$

Using the explicit form of f_0, (4.87), this can be simplified as

$$\tilde{f}_1(t) = -\sqrt{\sigma(t)} + f_1(t). \tag{4.104}$$

A comparison of f_1 from (4.90) with (4.57) confirms this result.

5 Zero modes of local operators in 2d CFT

In this section we show how to calculate zero modes of local operators in 2d CFT on a cylinder. Every qKdV charge is defined as such zero mode. So, it's essential to know how to calculate them effectively. Moreover, information about zero modes of quasi-primary operators will be essential when we will discuss Eigenstate Thermalization.

This section is organized as follows. In section 5.1 we develop general technique for calculating the zero modes of the normal ordered product of local operators. In section 5.2 we show in detail how this technique applies in the simplest case and calculate first non-trivial qKdV charge $\hat{Q}_3$. In section 5.3 we list the explicit expressions of the zero modes for all quasi-primary operators with the dimension of less or equal to eight. Then, in section 5.4 we give the explicit expression for $\hat{Q}_7$ in terms of the Virasoro algebra generators and verify its consistency in the large c limit and also using the constraints coming from the (9,2) minimal model.

5.1 Zero modes of the operator product

In this section we will describe general technique of finding zero modes of the product of arbitrary local operators $A(w)$ and $B(w)$ on a cylinder worldsheet, namely

$$(AB)_0 = \oint \frac{dw}{2\pi}(AB)(w), \tag{5.1}$$

where the contour is taken over a spacial circle. The operator $A(w)$ and $B(w)$ are assumed to have analytic mode expansion

$$A(w) = \sum_{n=-\infty}^{+\infty} A_n e^{-inw}, \tag{5.2}$$

and similarly for $B(w)$. The parentheses in $(AB)(w)$ denote normal ordering of the operators and is defined as

$$(AB)(w) = \oint_w \frac{dz}{2\pi i} \frac{1}{z-w} \mathcal{T}\left(A(z)B(w)\right), \tag{5.3}$$

where the integration is performed over the circle around w and the symbol $\mathcal{T}$ stands for "chronological ordering," i.e.

$$\mathcal{T}\left(A(z)B(w)\right) = \begin{cases} A(z)B(w), & \text{if } \operatorname{Im} z < \operatorname{Im} w, \\ B(w)A(z), & \text{if } \operatorname{Im} z > \operatorname{Im} w. \end{cases} \tag{5.4}$$

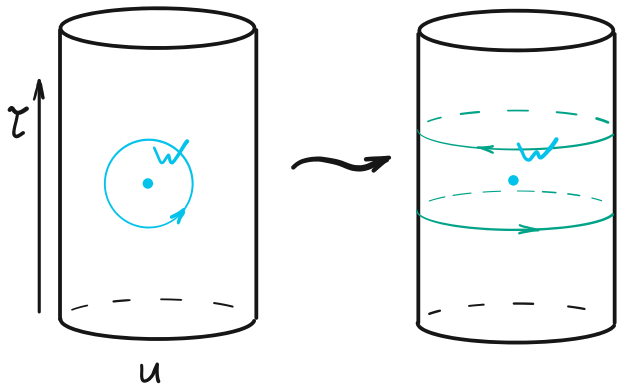

Figure 1. Deformation of the blue contour (5.3) into the green one (5.5).

To perform the integration (5.3) we will split the contour into two pieces as showed in the fugure 1, where we deform the contour in such a way that we can deal with two chronologically ordered expressions separately, namely

$$(AB)(w) = \int_{-i\epsilon}^{2\pi - i\epsilon} \frac{dz}{2\pi i} \frac{A(z)B(w)}{z-w} - \int_{i\epsilon}^{2\pi + i\epsilon} \frac{dz}{2\pi i} \frac{B(w)A(z)}{z-w}. \tag{5.5}$$

The strategy of calculating (5.5) is to express the integrand in terms of mode expansion e^{iw} and e^{iz}. Let us introduce two axillary variables u and v,

$$u = e^{iz}, \quad v = e^{iw}, \tag{5.6}$$

and make the following transformation,

$$\begin{aligned} \frac{1}{2\pi i}\frac{dz}{z-w} = \frac{1}{2\pi}\frac{dz}{\log(u)-\log(w)} = \frac{1}{2\pi}\frac{dz}{\log\left(1+\frac{u-v}{v}\right)} = \\ = \frac{dz}{2\pi}\left(\frac{v}{u-v} + \sum_{k=0}^{\infty} c_k \left(\frac{u-v}{v}\right)^k\right), \end{aligned} \tag{5.7}$$

where we formally expanded one over logarithm and c_k denote the coefficients of this expansion. The combination $v/(u-v)$ can be easily represented in terms of the mode expansion, namely

$$\frac{v}{u-v} = \begin{cases} \sum_{k=1}^{+\infty} e^{ik(w-z)}, & \mathrm{Im}(z) < \mathrm{Im}(w), \\ \sum_{k=0}^{+\infty} e^{ik(z-w)}, & \mathrm{Im}(z) > \mathrm{Im}(w). \end{cases} \tag{5.8}$$

Plugging (5.7) and (5.8) into (5.5) we obtain

$$\begin{aligned} (AB)(w) = \int_{-i\epsilon}^{2\pi-i\epsilon} \frac{dz}{2\pi}\left(\sum_{k=1}^{\infty} e^{ik(w-z)} + \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)}-1\right)^k\right) A(z)B(w) - \\ - \int_{+i\epsilon}^{2\pi+i\epsilon} \frac{dz}{2\pi}\left(-\sum_{k=0}^{\infty} e^{ik(z-w)} + \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)}-1\right)^k\right) B(w)A(z). \end{aligned} \tag{5.9}$$

In each of the two integrals in (5.9) we can integrate the first term assuming (5.2) and combine the rest into an integral of the commutator $[A(z), B(w)]$, i.e.

$$\begin{aligned} (AB)(w) = A_-(w)B(w) + B(w)A_+(w) + \\ + \int_0^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)}-1\right)^k [A(z), B(w)], \end{aligned} \tag{5.10}$$

where

$$A_-(w) = \sum_{n=1}^{\infty} A_{-n} e^{inw}, \quad A_+(w) = \sum_{n=0}^{\infty} A_n e^{-inw}. \tag{5.11}$$

Substituting (5.10) into (5.1) we obtain

$$\begin{aligned} (AB)_0 = \sum_{n=1}^{\infty} A_{-n}B_n + \sum_{n=0}^{\infty} B_{-n}A_n + \\ + \int_0^{2\pi} \frac{dw}{2\pi} \int_0^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)}-1\right)^k [A(z), B(w)]. \end{aligned} \tag{5.12}$$

To move further one should calculate the commutator $[A(z), B(w)]$ and perform the integration of an infinite sum over k. This seems to be a difficult task, but we will see shortly that only finite number of terms in this sum give non-zero contributions to the integral

$$\int_0^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1\right)^k [A(z), B(w)] \tag{5.13}$$

from (5.10). Indeed, if we fix k and expand $\left(e^{i(z-w)} - 1\right)^k$ we immediately see that the only modes of $A(z)$ which contribute are those from 0 to k. More precisely for a given fixed k we have

$$\sum_{n=0}^{k} c_k \frac{k!}{n!(n-k)!} (-1)^n e^{-inw} [A_n, B(w)] \tag{5.14}$$

If we now rewrite the commutator in terms of the mode expansion $e^{-inw}[A_n, B(w)] = \sum_m [A_n, B_m] e^{-(n+m)w}$ and keep in mind that both operators A and B are "built" out of stress-energy tensor, then both A_n and B_n will be some normal ordered polynomials in L_i such that the total sum of indexes is equal to n. The commutator $[A_n, B_m]$ is therefore also a polynomial in L_i with a coefficient which is a polynomial in n. Therefore the sum in (5.14) will be a linear combination of the terms

$$S(a, k) = \sum_{n=0}^{k} \frac{1}{n!(n-k)!} (-1)^{k-n} n^a, \tag{5.15}$$

where a is some non-zero integer. The expression above is the Stirling number of the second kind, which vanishes unless $k \leq a$. This immediately confirms that only finite number of terms in (5.14) with $k \leq a$ contribute.

We illustrate the emergence of the polynomial expression in n explicitly in the case $A = B = T(w)$ in the next section. Here we only note that the finite number of terms contributing in the sum over k in (5.10) provide a crucial simplification. It allows computing (5.10) efficiently using computer algebra, which we use extensively in the computation of the zero modes in section 5.3.2 as well as of Q_7 in section 5.4.

5.2 Warm-up: computation of $\hat{Q}_3$

In this section we apply the machinery devised in the previous section to the simplest non-trivial example $\hat{Q}_3 = \oint \frac{dz}{2\pi}(TT)(z)$ and show explicitly how to perform the integration (5.12). It's convenient to introduce shifted Virasoro generators,

$$\tilde{L}_n = L_n - \frac{c}{24}\delta_{n,0}. \tag{5.16}$$

In terms of these operators the stress-energy tensor $T(z)$ has the following mode expansion

$$T(z) = \sum_{n=-\infty}^{+\infty} \tilde{L}_n e^{-inz}, \tag{5.17}$$

and the Virasoro algebra in terms of the shifted generators is modified as

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}n^3\delta_{n+m,0}. \tag{5.18}$$

Substituting $A(z) = T(z)$ and $B(w) = T(w)$ into (5.12) we obtain

$$\begin{aligned} \hat{Q}_3 = (TT)_0 = 2\sum_{n=1}^{+\infty} \tilde{L}_{-n}\tilde{L}_n + \tilde{L}_0^2 + \\ + \int_0^{2\pi}\frac{dw}{2\pi}\int_0^{2\pi}\frac{dz}{2\pi}\sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1\right)^k [T(z), T(w)]. \end{aligned} \tag{5.19}$$

The calculation of the commutator is a bit tedious but straightforward:

$$\begin{aligned} [T(z), T(w)] = \sum_{m,n\in\mathbb{Z}} e^{-inz}e^{-imw}[\tilde{L}_n, \tilde{L}_m] = \\ = \sum_{m,n\in\mathbb{Z}} e^{-imz}e^{-inw}\left((n-m)\tilde{L}_{n+m} + \frac{c}{12}n^3\delta_{n+m,0}\right) = \\ = \sum_{m,n\in\mathbb{Z}} e^{-in(z-w)}(2n-(m+n))e^{-i(m+n)w}\tilde{L}_{m+n} + \frac{c}{12}\sum_{n\in\mathbb{Z}} n^3 e^{-in(z-w)}. \end{aligned} \tag{5.20}$$

As was anticipated in the end of previous subsection, the commutator of two local operators gives rise to a polynomial expression in n. Namely, using the notations of the previous section (A_n is equal to $\tilde{L}_n$ and $B(w)$ is equal to $T(w)$),

$$e^{-inw}[A_n, B(w)] = \sum_{m\in\mathbb{Z}}(2n-m)e^{-imw}\tilde{L}_m + \frac{c}{12}n^3,$$

and therefore only terms with $k \leq 3$ will contribute in the sum over k in (5.10). Here we would like to illustrate that by combing (5.20) into a local expression. We continue,

$$\begin{aligned} [T(z), T(w)] = \\ 2(i\partial_z)\sum_{n\in\mathbb{Z}} e^{-in(z-w)}T(w) + i\sum_{n\in\mathbb{Z}} e^{-in(z-w)}\partial_w T(w) + \frac{c}{12}(i\partial_z)^3\sum_{n\in\mathbb{Z}} e^{-in(z-w)} = \\ = 4\pi i\partial_z\delta(z-w)T(w) + 2\pi i\delta(z-w)\partial_w T(w) + \frac{c}{12}(i\partial_z)^3 2\pi\delta(z-w), \end{aligned} \tag{5.21}$$

where on the last line we have used the following representation of delta-function,

$$2\pi\delta(z-w)=\sum_{n\in\mathbb{Z}}e^{-in(z-w)}. \tag{5.22}$$

Note, that we have represented the commutator of stress-energy tensors $[T(z),T(w)]$ in such a way that every term in the final expression contains delta-function $\delta(z-w)$ or its derivatives. This representation helps us easily perform the integration over z in (5.19), namely

$$\oint\frac{dz}{2\pi}\sum_{k=0}^{\infty}c_k\left(e^{i(z-w)}-1\right)^k[T(z),T(w)]=ic_0\partial_wT(w)- \\ -2i\partial_z\sum_{k=0}^{+\infty}c_k\left(e^{i(z-w)}-1\right)^k\Big|_{z=w}T(w)+\frac{c}{12}\left(-i\partial_z\right)^3\sum_{k=0}^{+\infty}c_k\left(e^{i(z-w)-1}\right)^k\Big|_{z=w}. \tag{5.23}$$

Here in the first term we have used $0^0=1$. Let us denote

$$a_k=-i\partial_z\left(e^{i(z-w)}-1\right)^k\Big|_{z=w},\quad b_k=\left(-i\partial_z\right)^3\left(e^{i(z-w)}-1\right)^k\Big|_{z=w}. \tag{5.24}$$

The coefficients a_k and b_k are non-zero only for the first few values of $k\leq 3$. Specifically,

$$a_1=1,\quad a_{2,3,4,\ldots}=0, \\ b_1=1,\quad b_2=6,\quad b_3=6,\quad b_{4,5,6,\ldots}=0. \tag{5.25}$$

That means that the sums $\sum c_k\,a_k$ and $\sum c_k\,b_k$ contain only finite number of terms and only four first coefficients c_k contribute to these sums. From the definition we find

$$c_0=\frac{1}{2},\quad c_1=-\frac{1}{12},\quad c_2=\frac{1}{24},\quad c_3=-\frac{19}{720}. \tag{5.26}$$

Combining (5.23) and (5.10) we get the normal ordered expression

$$(TT)(w)=T_-(w)T(w)+TT_+(w)-\frac{1}{6}T(w)+\frac{i}{2}\partial_wT(w)+\frac{c}{1440}, \tag{5.27}$$

where $T_-(w)=\sum_{k=1}e^{ikw}\tilde{L}_{-k}$ and $T_+(w)=\sum_{k=0}e^{-ikw}\tilde{L}_k$.

And, finally, for zero mode

$$\hat{Q}_3=(TT)_0=\oint\frac{dw}{2\pi}(TT)(w)=2\sum_{n=1}^{+\infty}\tilde{L}_{-n}\tilde{L}_n+\tilde{L}_0^2-\frac{1}{6}\tilde{L}_0+\frac{c}{1440}= \\ =2\sum_{n=1}^{+\infty}L_{-n}L_n+L_0^2-\frac{c+2}{12}L_0+\frac{c(5c+22)}{2880}. \tag{5.28}$$

The result matches that one of [18]. The technique we have described here in principle can be applied to a calculation of zero modes of any product of local operators.

5.3 Quasi-primaries

In this section we list the explicit expressions for the zero modes of all quasi-primaries from the vacuum family with the dimension less or equal to eight. Up to dimension nine all quasi-primaries have even dimension. There is a unique operator of dimension zero – the identity operator, which is a primary. There is also a unique quasi-primary at level two, the stress-energy tensor T. Its zero mode is the CFT Hamiltonian – first KdV charge $\hat{Q}_1 = L_0 - c/24$. At level two there is also a unique quasi-primary

$$T_2 = (TT) - \frac{3}{10}\partial^2 T. \tag{5.29}$$

Its zero mode is the first non-trivial KdV charge $\hat{Q}_3$ given by (5.28). At all other levels the quasi-primaries are not unique and we organize them by dimension, nested powers of T and orthogonality of Zamolodchikov metric.

5.3.1 Quasi-primaries of dimension 6

At level four there are two quasi-primaries,

$$\mathcal{B} = (\partial T \partial T) - \frac{4}{5}(\partial^2 TT) - \frac{1}{42}\partial^4 T, \tag{5.30}$$

and

$$\mathcal{D} = (T(TT)) - \frac{9}{10}(\partial^2 TT) - \frac{1}{28}\partial^4 T + \frac{93}{70c+29}\mathcal{B}. \tag{5.31}$$

Zero mode of their combination $\mathcal{D} - \frac{5(43+14c)}{2(29+70c)}\mathcal{B}$ is the KdV charge $\hat{Q}_5$, which was found explicitly in [18]. To find the explicit form of $\mathcal{B}_0$ and $\mathcal{D}_0$ we introduce the "building block"

$$(\partial T \partial T)_0 = -(\partial^2 TT)_0 = 2\sum_{n=1}^{\infty} n^2 \tilde{L}_{-n}\tilde{L}_n + \frac{\tilde{L}_0}{60} - \frac{c}{3024}, \tag{5.32}$$

which is different from the quasi-primary $\mathcal{B}$ by a total derivative. Therefore

$$\mathcal{B}_0 = \frac{9}{5}(\partial T \partial T)_0. \tag{5.33}$$

Similarly, to calculate $\mathcal{D}_0$ we introduce

$$\begin{aligned}(T(TT))_0 = \sum_{k,l=0}^{\infty} \tilde{L}_{-k-l}\tilde{L}_k\tilde{L}_l + 2\sum_{k=1,l=0}^{\infty} \tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_l + \sum_{k,l=1}^{\infty} \tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} - \\ - \sum_{n=1}^{\infty} \tilde{L}_{-n}\tilde{L}_n - \frac{\tilde{L}_0^2}{2} + \frac{c\tilde{L}_0}{480} + \frac{\tilde{L}_0}{15} - \frac{c}{3024},\end{aligned} \tag{5.34}$$

such that

$$\mathcal{D}_0 = (T(TT))_0 + \frac{9}{10}(\partial T \partial T)_0. \tag{5.35}$$

5.3.2 Quasi-primaries of dimension 8

At level eight there are three quasi-primaries,

$$\mathcal{E} = (\partial^2 T \partial^2 T) - \frac{10}{9}(\partial^3 T \partial T) + \frac{10}{63}(\partial^4 TT) - \frac{1}{324}\partial^6 T, \tag{5.36}$$

$$\mathcal{H} = (\partial T(\partial TT)) - \frac{4}{5}(\partial^2 T(TT)) + \frac{2}{15}(\partial^3 T \partial T) - \frac{3}{70}(\partial^4 TT) + \frac{9(140c+83)}{50(105c+11)}\mathcal{E} \tag{5.37}$$

and

$$\begin{aligned}\mathcal{I} = (T(T(TT))) - \frac{9}{5}(\partial^2 T(TT)) + \frac{3}{10}(\partial^3 T \partial T) + \frac{81(35c-51)}{100(105c+11)}\mathcal{E} + \\ + \frac{12(465c-127)}{5c(210c+661)-251}\mathcal{H}.\end{aligned} \tag{5.38}$$

The "building block" for calculating $\mathcal{E}_0$ is

$$(\partial^2 T \partial^2 T)_0 = -(\partial^3 T \partial T)_0 = (\partial^4 TT)_0 = 2\sum_{n=1}^{\infty} n^4 \tilde{L}_{-n}\tilde{L}_n - \frac{\tilde{L}_0}{126} + \frac{c}{2880}, \tag{5.39}$$

And therefore,

$$\mathcal{E}_0 = \frac{143}{63}(\partial^2 T \partial^2 T)_0. \tag{5.40}$$

There are two "building blocks" for $\mathcal{H}$:

$$\begin{aligned}(\partial T(\partial TT))_0 = -\sum_{k=0,l=0}^{\infty} kl\tilde{L}_{-k-l}\tilde{L}_k\tilde{L}_l + 2\sum_{k=1,l=0}^{\infty} kl\tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_l - \\ - \sum_{k,l=1}^{\infty} kl\tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} + \frac{1}{12}(\partial T \partial T)_0 + \frac{1}{30}\sum_{n=1}^{\infty}\tilde{L}_{-n}\tilde{L}_n + \frac{\tilde{L}_0^2}{60} - \frac{(5c+93)\tilde{L}_0}{15120} + \frac{113c}{1814400}\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}(\partial^2 T(TT))_0 = -\sum_{k,l=0}^{\infty} l^2\tilde{L}_{-k-l}\tilde{L}_k\tilde{L}_l - \sum_{k=1,l=0}^{\infty}(k^2+l^2)\tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_l - \\ - \sum_{k=1,l=1}^{\infty} k^2\tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} + \frac{7}{120}\sum_{n=1}^{\infty}\tilde{L}_{-n}\tilde{L}_n - \frac{\tilde{L}_0^2}{30} + \frac{c\tilde{L}_0}{1512} + \frac{61\tilde{L}_0}{7560} - \frac{131c}{1814400}.\end{aligned} \tag{5.42}$$

The zero mode of $\mathcal{H}$ is

$$\mathcal{H}_0 = (\partial T(\partial TT))_0 - \frac{4}{5}(\partial^2 T(TT))_0 + \frac{7035c+13652}{110250c+11550}(\partial^2 T \partial^2 T)_0. \tag{5.43}$$

The expression for $(T(T(TT)))_0$ is too bulky to write it twice. We do not write it explicitly here, but simply mention that it can be obtained from the Virasoro algebra expression for $\hat{Q}_7$, which we give explicitly in the next section, by subtracting $(\partial^2 T \partial^2 T)_0$ and $(T(\partial T \partial T))_0$ with proper coefficients, see equation (5.46). Thus, in lieu of $(T(T(TT)))_0$ we give explicitly the expression for $(T(\partial T \partial T))_0$,

$$\begin{aligned}(T(\partial T\partial T))_0 = &\sum_{k,l=1}^{\infty}(k+l)l\tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} + \sum_{l=0,k=1}^{\infty}(k-l)k\tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_l + \\ &+\sum_{k,l=0}^{\infty}(k+l)k\tilde{L}_{-k-l}\tilde{L}_k\tilde{L}_l + \sum_{k=0,l=1}^{\infty}(k-l)k\tilde{L}_{-l}\tilde{L}_{l-k}\tilde{L}_k - \\ &-\frac{7}{6}\sum_{n=1}^{\infty}n^2\tilde{L}_{-n}\tilde{L}_n + \frac{1}{30}\sum_{n=1}^{\infty}\tilde{L}_{-n}\tilde{L}_n + \frac{\tilde{L}_0^2}{60} + \frac{c\tilde{L}_0}{3024} - \frac{\tilde{L}_0}{135} + \frac{13c}{86400}.\end{aligned} \tag{5.44}$$

Finally, for the last quasi-primary at level 8 we have

$$\begin{aligned}\mathcal{I}_0 = (T(T(TT)))_0 - \frac{9}{5}(\partial^2 T(TT))_0 - \frac{3}{10}(\partial^2 T\partial^2 T)_0 + \frac{81(35c-51)}{100(105c+11)}\mathcal{E}_0 + \\ + \frac{12(465c-127)}{5c(210c+661)-251}\mathcal{H}_0.\end{aligned} \tag{5.45}$$

5.4 Expression for $\hat{Q}_7$

In this section we present the expression for $\hat{Q}_7$ in terms of Virasoro generators, which we calculated by applying the technique described above. Then we test our result by showing that it is consistent with the known spectrum $\hat{Q}_7$ at the leading $1/c$ order. We further check at that our expression for $\hat{Q}_7$ vanishes as an operator for the $(9,2)$ minimal model at the first dozen of descendant levels, as is predicted in [18, 34]. Finally, we will show how to use commutativity of qKdV charges, known results about spectrum in $1/c$ expansion and the constraints from the minimal models to get a shortcut for the expression (5.47).

5.4.1 The result

$\hat{Q}_7$ is the zero mode of the operator [34]

$$J_8 = (T(T(TT))) + \frac{c+2}{3}(T(\partial T\partial T)) + \frac{2c^2-17c-42}{360}(\partial^2 T\partial^2 T). \tag{5.46}$$

In principle J_8 (and higher densities J_{2n}) can be determined by requiring commutativity of $\hat{Q}_7$ with $\hat{Q}_5$ and $\hat{Q}_3$, however, the calculation is quite involved. Alternatively, one can use the expression for the thermal correlation function $\langle \hat{Q}_{2n-1} \rangle$, which can be calculated

using other means and fix the coefficients of J_{2n} in this way. This was done for J_8, J_{10} and J_{12} in [34]. However, for J_{14} and higher densities the number of independent coefficients becomes too large to be uniquely fixed from the form of $\langle \hat{Q}_{2n-1} \rangle$ alone.

We find the following expression for $\hat{Q}_7$ in terms of the Virasoro algebra generators

$$\begin{aligned}
\hat{Q}_7 = &\sum_{k,l,m=1}^{\infty} L_{-k}L_{-l}L_{-m}L_{k+l+m} + \sum_{k,l,m=0}^{\infty} L_{-k-l-m}L_kL_lL_m + \\
&+3\sum_{\substack{k,l=1\\m=0}}^{\infty} L_{-k}L_{-l}L_{k+l-m}L_m + 3\sum_{\substack{k=1\\l,m=0}}^{\infty} L_{-k}L_{k-l-m}L_lL_m + \\
&\frac{8+c}{3}\left[\sum_{k,l=1}^{\infty}(k+l)lL_{-k}L_{-l}L_{k+l} + \sum_{\substack{k=1\\l=0}}^{\infty}(k-l)kL_{-k}L_{k-l}L_l\right] + \\
&+\frac{8+c}{3}\left[\sum_{k,l=0}^{\infty}(k+l)kL_{-k-l}L_kL_l + \sum_{\substack{k=0\\l=1}}^{\infty}(k-l)kL_{-l}L_{l-k}L_k\right] + \\
&+\sum_{n=1}^{\infty}\left(\frac{c^2-c-141}{90}n^4 - \frac{7c+59}{18}n^2\right)L_{-n}L_n - \left(\frac{1}{48}c^2 + \frac{53}{360}c + \frac{19}{90}\right)\tilde{Q}_3 - \\
&-\left(\frac{1}{6}c+1\right)\tilde{Q}_5 - \frac{c+6}{6}L_0^3 + \frac{15c^2+194c+568}{1440}L_0^2 - \\
&-\frac{(c+2)(c+10)(3c+28)}{10368}L_0 + \frac{c(3c+46)(25c^2+426c+1400)}{24883200}.
\end{aligned} \tag{5.47}$$

This is one of the main results of this section. Here $\tilde{Q}_3$ and $\tilde{Q}_5$ are defined as parts of $\hat{Q}_3$ and $\hat{Q}_5$ which annihilate the primary states, $\tilde{Q}_3 |\Delta\rangle = 0$ and $\tilde{Q}_5 |\Delta\rangle = 0$, namely,

$$\tilde{Q}_3 = 2\sum L_{-n}L_n, \tag{5.48}$$

and

$$\begin{aligned}
\tilde{Q}_5 = &\sum_{k,l=0}^{\infty} L_{-k-l}L_kL_l + 2\sum_{k=1,l=0}^{\infty} L_{-k}L_{k-l}L_l + \sum_{k,l=1}^{\infty} L_{-k}L_{-l}L_{k+l} + \\
&+\sum_{n=1}^{\infty}\left(\frac{c+2}{6}n^2 - \frac{c}{4} - 1\right)L_{-n}L_n - L_0^3.
\end{aligned} \tag{5.49}$$

The expression (5.49) can also be represented in a slightly different way [18] using the

following identity,

$$\sum_{k,l=0}^{\infty} L_{-k-l}L_kL_l + 2\sum_{k=1,l=0}^{\infty} L_{-k}L_{k-l}L_l + \sum_{k,l=1}^{\infty} L_{-k}L_{-l}L_{k+l} = \sum_{n_1+n_2+n_3=0} :L_{n_1}L_{n_2}L_{n_3}: + \frac{3}{2}\sum_{n=1} n^2 L_{-n}L_n + \frac{3}{2}\sum_{r=1} L_{1-2r}L_{2r-1}. \tag{5.50}$$

5.4.2 The consistency check: $1/c$ expansion and the (9, 2) minimal model

In this section we perform two different consistency checks of the Virasoro algebra expression for $\hat{Q}_7$ (5.47).

In section 3 the spectrum of all qKdV charges was calculated using semi-classical quantization at the first few orders in $1/c$ expansion. This calculation did not rely on the explicit form of $\hat{Q}_{2k-1}$ in terms of the Virasoro algebra generators, and hence can be used to cross-check our result.

Leading $1/c$ spectrum of $\hat{Q}_7$ in terms of "free bosons" n_k is given by (3.39) $\hat{Q}_7|\lambda\rangle = \lambda|\lambda\rangle$, where

$$\begin{aligned}
\lambda_7 = {}& \Delta'^4 + \Delta'^3\left(28\sum_k n_k k - 1\right) + \Delta'^2 c\left(\frac{7}{3}\sum_k n_k k^3 + \frac{7}{720}\right) + \\
&+\Delta' c^2\left(\frac{7}{90}\sum_k n_k k^5 - \frac{1}{6480}\right) + c^3\left(\frac{1}{1080}\sum_k n_k k^7 + \frac{1}{518400}\right) + \\
&+\Delta'^2\left(98n^2 - 77\sum_k n_k^2 k^2 + \frac{259}{3}\sum_k n_k k^3 - 77\sum_k n_k k^2 - \frac{14}{3}\sum_k n_k k + \frac{71}{180}\right) + \\
&+\Delta' c\left(\frac{98}{15}\sum_{k,l} k^3 l n_k n_l - \frac{56}{15}\sum_k n_k^2 k^4 + \frac{63}{25}\sum_k n_k k^5 - \right. \\
&\left. -\frac{56}{15}\sum_k n_k k^4 - \frac{7}{90}\sum_k n_k k^3 + \frac{49}{1800}\sum_k n_k k - \frac{23}{4320}\right) + \\
&+c^2\left(\frac{7}{180}\sum_{k,l} n_k n_l k^3 l^3 + \frac{7}{90}\sum_{k,l} n_k n_l k^5 l - \frac{7}{120}\sum_k n_k^2 k^6 + \frac{127}{5400}\sum n_k k^7 \right. \\
&\left. -\frac{7}{120}\sum_k n_k k^6 + \frac{7}{21600}\sum_k n_k k^3 - \frac{1}{6480}\sum_k n_k k + \frac{103}{2073600}\right) - \\
&-504\frac{\Delta'^3}{c}\left(\sum_k n_k^2 - 2\sum_k n_k k + \sum_k n_k\right) + O(c)
\end{aligned} \tag{5.51}$$

Using computer algebra one can check explicitly at first dozen of descendant levels that this spectrum matches with the one following from (5.47).

Another check is provided by [18, 34], which shows that for the minimal models $(2n+3,2)$, $n \leqslant 1$, the qKdV charges Q_k with k divisible by $2n+1$ vanish as operators. Hence Q_7 should vanish as an operator in the minimal model $(9,2)$, with $n=3$ and central charge $c=-46/3$. This minimal model includes primaries with the following dimensions $\{\Delta_k\} = \{0,-1/3,-2/3,-5/9\}$. Using computer algebra we have verified that Q_7 vanishes for all non-zero states of this model up to the descendant level twelve.

5.4.3 A shortcut

In this subsection we show how to get the Virasoro algebra expression for $\hat{Q}_7$ without full explicit calculation of all involved commutators by exploiting the restrictions from the commutativity of qKdV charges, $1/c$ expansion and the $(9,2)$ minimal model. Our goal here will be to understand what kind of terms may appear in the Virasoro algebra expression of $\hat{Q}_7$ and then fix the coefficients using the constraints.

We start from the last term of J_8 (5.46):

$$A_8 = (\partial^2 T \partial^2 T). \tag{5.52}$$

Substiting A_8 into (5.10 - 5.12) we obtain

$$\oint \frac{dw}{2\pi} A_8 \sim \sum_{n=1}^{\infty} n^4 L_{-n} L_n + comm, \tag{5.53}$$

where *comm* comes from the integral of the commutator in (5.10). Expressing the commutator in terms of Virasoro generators we get $[\partial^2 T, \partial^2 T] \sim [L_n, L_m] \sim L_k$ due to Virasoro algebra. It is easy to see that the final answer for the zero mode should contain only such operators that map states of level k to the states of level k. That means that after integrating the commutator we can only get some function of L_0. Therefore,

$$\oint \frac{dw}{2\pi} A_8 \sim \sum_{n=1}^{\infty} n^4 L_{-n} L_n + f(L_0). \tag{5.54}$$

Now we turn to the next term

$$B_8 = (T(\partial T \partial T)). \tag{5.55}$$

The term B_8 contains nested normal ordering. We will deal with it subsequently,

$$(\partial T \partial T)(w) = \partial T \partial T_+ + \partial T_- \partial T + \oint \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1\right)^k [\partial T(z), \partial T(w)]. \tag{5.56}$$

One can calculate the commutator and perform the integration (5.56) explicitly but we just notice that the result of the integration can only contain terms with one stress-energy tensor T or its derivatives and the terms like TT or $T\partial T$ are not present due to Virasoro algebra, which means

$$(\partial T\partial T)(w) = \partial T(w)\partial T_+(w) + \partial T_-(w)\partial T(w) + f(T(w), \partial T(w), \partial^2 T(w)), \quad (5.57)$$

where $f(T, \partial T, \partial^2 T)$ is some linear function of its arguments. Substituting (5.56) into B_8 we get

$$(T(\partial T\partial T)) = T_-\partial T\partial T_+ + \partial T\partial T_+T_+ + T_-\partial T_-\partial T + \partial T_-\partial TT_+ + comm. \quad (5.58)$$

Here *comm* again denotes some expression associated with the commutators, which we will not calculate explicitly. But let us notice again that it contains at most two stress-tensors or its derivatives. The only expression that survives after integration of such terms is proportional to $\sim L_{-n}L_n$, namely

$$\oint dz\partial^\alpha T\partial^\beta T \sim \sum_{-\infty}^{+\infty} n^{\alpha+\beta} L_{-n}L_n. \quad (5.59)$$

Counting the amount of dervatives and integrating (5.58) we obtain

$$\begin{aligned}\oint \frac{dw}{2\pi} B_8 = &\sum_{k,l=1}^{\infty}(k+l)lL_{-k}L_{-l}L_{k+l} + \sum_{\substack{k=1\\l=0}}^{\infty}(k-l)kL_{-k}L_{k-l}L_l + \\ &+ \sum_{k,l=0}^{\infty}(k+l)kL_{-k-l}L_kL_l + \sum_{\substack{k=0\\l=1}}^{\infty}(k-l)kL_{-l}L_{l-k}L_k + \\ &+ \sum_{n=1}^{\infty}\left(\alpha n^4 + \beta n^2 + \gamma\right)L_{-n}L_n + f(L_0),\end{aligned} \quad (5.60)$$

where α, β, γ are some c-dependent coefficients and $f(L_0)$ is some function of c and L_0.

In the same manner one can deal with the term $(T(T(TT))$. As a result we get an

expression for $\hat{Q}_7$ with some coefficients to fix. The expression is

$$\begin{aligned}\hat{Q}_7 = &\sum_{k,l,m=1}^{\infty} L_{-k}L_{-l}L_{-m}L_{k+l+m} + \sum_{k,l,m=0}^{\infty} L_{-k-l-m}L_kL_lL_m + \\ &+3\sum_{\substack{k,l=1\\m=0}}^{\infty} L_{-k}L_{-l}L_{k+l-m}L_m + 3\sum_{\substack{k=1\\l,m=0}}^{\infty} L_{-k}L_{k-l-m}L_lL_m + \\ &\frac{8+c}{3}\left[\sum_{k,l=1}^{\infty}(k+l)lL_{-k}L_{-l}L_{k+l} + \sum_{\substack{k=1\\l=0}}^{\infty}(k-l)kL_{-k}L_{k-l}L_l\right] +, \\ &+\frac{8+c}{3}\left[\sum_{k,l=0}^{\infty}(k+l)kL_{-k-l}L_kL_l + \sum_{\substack{k=0\\l=1}}^{\infty}(k-l)kL_{-l}L_{l-k}L_k\right] + \\ &+\sum_{n=1}^{\infty}\left(\alpha n^4+\beta n^2\right)L_{-n}L_n + \gamma\tilde{Q}_3 + \delta\tilde{Q}_5 + f(L_0)\end{aligned} \tag{5.61}$$

where α, β, γ and δ are the coefficients dependent on central charge and $f(L_0)$ is some polynomial of L_0 and central charge.

The term $f(L_0)$ determines the value of $\hat{Q}_7$ on a primary state. That has been previously calculated in the appendix B of [22]. The coefficients α and β can be found from the commutativity constraint $[\hat{Q}_3, \hat{Q}_7] = 0$. From $1/c$ expansion of the spectrum (5.51) we see that δ is at most linear polynomial in c, $\delta = \delta_1 c + \delta_2$ and γ is at most quadratic polynomial in c, $\gamma = \gamma_1 c^2 + \gamma_2 c + \gamma_3$. The coefficients δ_1, δ_2 and γ_1, γ_2 can be extracted directly from (5.51) and, finally, γ_3 can be fixed by requiring that Q_7 vanishes acting on any descendant state of the (9,2) minimal model.

In this section we pedagogically developed and presented the machinery of calculating the zero modes of local operators in a 2d CFT on a cylinder. We focused on the situation when the operators are from the vacuum family, i.e. are built from the powers of stress-energy tensor and its derivatives. The explicit formulae obtained in this section will be used in section 7 when will study spectral properties of the qKdV hierarchy and Generalized Eigenstate Thermalization Hypothesis in 2d CFTs.

6 ETH and 2d CFT

In this section we discuss Eigenstate Thermalization Hypothesis (ETH) and its applicability to 2d CFT. We start with brief introduction to a conventional formulation of ETH for general quantum many-body systems in section 6.1. Then we discuss how

to formulate the hypothesis specifically for 2d CFTs, focusing on Virasoro descendants of the "identity block" (6.2). In section 6.3 we show how ETH works in the limit of infinite central charge $c \to \infty$. In section 6.4 we outline non-perturbative nature of ETH for sufficiently highly excited states.

6.1 Eigenstate Thermalization Hypothesis

During the last two decades there has been significant progress in understanding how quantum statistical physics emerges from the dynamics of an isolated quantum many-body system.

Eigenstate Thermalization Hypothesis (ETH), developed in the works of Deutsch and Srednicki in the 90s [1, 2] is the idea that individual energy eigenstate of a sufficiently complex "chaotic" many-body quantum system exhibits thermal properties. ETH provides a mechanism explaining thermalization of isolated quantum systems. During last decade extensive numerical studies have supported the expectation that ETH is a general property of quantum many-body systems, unless the system exhibits an extensive number of conserved quantities [3].

At the colloquial level the ETH promotes a highly excited energy eigenstate of a quantum system to the "eigenstate ensemble" stating that the latter can describe thermal properties of a quantum system in the same way as the conventional canonical and micro-canonical ensembles. The discrepancy between e.g. expectation values of local quantities in these different ensembles then would be suppressed in the thermodynamic limit. Quantitatively, for a few-body operator $\mathcal{O}$, ETH postulates that

$$\langle E_i | \mathcal{O} | E_j \rangle = f_{\mathcal{O}}(E)\delta_{ij} + \Omega^{-1/2}(E) r_{ij}, \quad E = (E_i + E_j)/2. \tag{6.1}$$

Here $|E_i\rangle$ denotes an energy eigenstate, $f_{\mathcal{O}}$ is a smooth function of E a $\Omega(E) = e^{S(E)}$ is the density of states of the full system. r_{ij} is a fluctuation matrix of order one. For a general quantum many-body system entropy $S(E)$ scales extensively with the volume V of a system, which means that in thermodynamic limit $V \to \infty$ the statement (6.1) simplifies to

$$\langle E_i | \mathcal{O} | E_i \rangle = f_{\mathcal{O}}(E_i). \tag{6.2}$$

In this limit a simple consequence of equivalence of ensembles gives

$$f_{\mathcal{O}}(E_i) = \frac{1}{Z(\beta)} \operatorname{Tr}\left(\mathcal{O} e^{-\beta H}\right). \tag{6.3}$$

6.2 ETH in 2d CFTs

It is not clear which CFTs can be considered as "chaotic". There should be enough of "degrees of freedom" for some form of thermalization to occur. In 2d CFTs central

charge c plays the role of "degrees of freedom". So, one can expect that $c \gg 1$ CFTs could thermalize.

Holographic CFTs are expected to be complex enough to exhibit thermalization starting from a sufficiently excited pure state. On the dual gravity side this is a process of black hole formation via gravitational collapse. Consequently holographic CFTs are expected to exhibit eigenstate thermalization, at least in some form,

$$\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(E_i) \tag{6.4}$$

with $f_{\mathcal{O}}$ being smooth. Operator-state correspondence allows reformulation (6.4) in terms of OPE coefficients. Then it can be easily shown that the primary states and the descendants can not both satisfy (6.4) with the same function $f_{\mathcal{O}}$.

It was proposed in [35] that the non-trivial content of ETH is not the equivalence between individual eigenstates and thermal ensemble, but the equivalence of individual eigenstates from a certain class with each other. Then the equivalence with the thermal ensemble would follow automatically, provided the original class of eigenstates is wide enough. This idea lead [36] to propose the following formulation of local ETH in CFTs: (6.4) should apply to any local operator $\mathcal{O}$ but be limited only to primary (Virasoro-primary in 2d) states $|E\rangle$. In this formulation $f_{\mathcal{O}}$ is not necessarily related to thermal expectation value of $\mathcal{O}$. For conformal theories in $d \geq 3$ it was further argued in [37] that the primary states dominate the microcanonical ensemble and therefore in the thermodynamic limit $f_{\mathcal{O}}$ coincides with the conventional thermal expectation value of $\mathcal{O}$. The $d = 2$ case is a subject of discussion below.

In 2d we need to distinguish two cases: when $\mathcal{O}$ is a Virasoro descendant of identity or not. In the former case the local ETH (6.4) is automatic – the corresponding heavy-heavy-light OPE coefficient is fixed by Virasoro algebra and is a smooth function of E, dimension of the Virasoro primary state $|E\rangle$. In the latter case, when $f_{\mathcal{O}}$ is not a descendant of identity, there are no known explicit examples when (6.4) is satisfied with a smooth $\mathcal{O}_{\text{eth}}(\beta)$. There is a general expectation though that this is the case for certain large central charge theories, including holographic CFTs. In $d = 2$ thermal expectation value of any such $\mathcal{O}$ is zero (because thermal cylinder is conformally flat). It is therefore often assumed that in sufficiently complex 2d theories corresponding heavy-heavy-light OPE is suppressed by the dimension of the heavy operator. Dominance of the vacuum conformal family or the "identity block" in the OPE of two heavy primaries is an underlying assumption in many works on large central charge theories in the context of ETH and thermalization

Leaving aside the behavior of $\mathcal{O}$ outside of vacuum conformal family, below we focus on the case when $\mathcal{O}$ is a Virasoro descendant of identity. Local ETH (6.4) is automatic

in this case, but function $f_{\mathcal{O}}$ a priory has no interpretation in terms of thermal physics. A natural question then would be to compare $f_{\mathcal{O}}$ with thermal expectation value of the operator $\mathcal{O}$,

$$\mathcal{O}_{\rm th}(\beta) = \frac{1}{Z(\beta)}{\rm Tr}(\mathcal{O}\, e^{-\beta H}). \tag{6.5}$$

It was expected that at infinite central charge locally eigenstate is equivalent to the thermal ensemble [38], which is indeed the case, $f_{\mathcal{O}} = \mathcal{O}_{\rm th}$. At the same time $f_{\mathcal{O}}$ and $\mathcal{O}_{\rm th}$ do not match at the subleading order in $1/c$ [37, 39–42]. A possible explanation of this discrepancy is that the "eigenstate ensemble" $|E\rangle$ has positive value of qKdV charges and hence should be compared not with (6.5) but with the full Generalized Gibbs Ensemble

$$\mathcal{O}_{\rm GGE}(\beta, \mu_i) = \frac{1}{Z(\beta, \mu_i)}{\rm Tr}(\mathcal{O}\, e^{-\beta H - \sum_k \mu_{2k-1}\hat{Q}_{2k-1}}), \tag{6.6}$$

with the fugacities μ_i chosen to match quantum numbers of $|E\rangle$. This is the comparison performed below.

6.3 Comparison at infinite central charge

From now on we restrict to the case when $|E\rangle$ is a heavy *scalar* Virasoro primary. To achieve finite energy density and thus finite effective temperature in the thermodynamic limit the dimension E should scale as ℓ^2,

$$\ell \to \infty, \qquad E/\ell^2 = \text{fixed}. \tag{6.7}$$

A Virasoro descendant of identity $\mathcal{O}$ can be either a quasi-primary or a total derivative. In the latter case by Lorentz invariance $f_{\mathcal{O}} = \mathcal{O}_{\rm GGE} = 0$. When $\mathcal{O}$ is a quasiprimary of dimension $2k$, $\mathcal{O}_{\rm eth}$ is non-trivial in the thermodynamic limit (6.7) only if $\mathcal{O}$ includes $\underbrace{(T\dots(TT))}_{\text{k times}}$. One can always choose a basis at the level $2k$ such that a unique quasi-primary with non-vanishing expectations value (6.4) in the thermodynamic limit is the density of KdV charge $\mathcal{O} = J_{2k}$ [37],

$$\hat{Q}_{2k-1} \equiv \int_0^\ell du\, J_{2k}. \tag{6.8}$$

To investigate the equality of the eigenstate and generalized Gibbs ensembles we compare

$$\langle E|J_{2k}|E\rangle = \frac{1}{\ell}\langle E|\hat{Q}_{2k-1}|E\rangle = \left(\frac{E}{\ell^2}\right)^k, \qquad E, \ell \to +\infty, \tag{6.9}$$

and

$$\langle J_{2k}\rangle_{GGE} = \frac{1}{\ell}\langle \hat{Q}_{2k-1}\rangle_{GGE} \equiv -\frac{1}{\ell}\partial_{\mu_{2k-1}} \log Z, \tag{6.10}$$

for $k \geq 2$. For $k = 1$ the equality between $\langle E|T|E\rangle$ and $\langle T\rangle_{GGE}$,

$$E = \langle H\rangle_{GGE} \tag{6.11}$$

is the relation which defines effective temperature β in terms of E/ℓ^2. The expectation values (6.9) satisfy $\langle E|J_{2k}|E\rangle = \langle E|J_2|E\rangle^k$, $q_1 \equiv T$. Therefore for the equality between the eigenstate and generalized Gibbs ensembles to hold it is necessary for the GGE partition function to satisfy (4.26). This is the case at leading order in central charge for any values of μ_{2k-1} as discussed in section 4.2. Hence we establish that for the descendant of identity $\mathcal{O} = J_2$, at leading order in central charge expectation value in a primary state is the same as in the generalized Gibbs ensemble for any choice of μ_{2k-1}. The same conclusion, about the equivalence of the eigenstate and the GGE for any choice of μ_{2k-1} also trivially applies to all other descendants of identity, because for them at leading order in central charge both (6.4) and (7.40) are zero. Equivalence for all μ_{2k-1} is consistent with the holographic interpretation that at infinite central charge classical black hole in AdS_3 is dual to (4.1) for any values of μ_{2k-1} [43]. Also see [44] for bulk interpretation of the KdV charges in terms of boundary gravitons.

6.4 Discrepancy at finite central charge

At finite central charge the relations (4.26) are not automatically satisfied. Using the expansion (4.31), at leading order in $1/c$ one finds from (4.27,4.28),

$$2(f_0 + 3t_3\partial_{t_3}f_0 + 5t_5\partial_{t_5}f_0 + \dots)(f_1 + 3t_3\partial_{t_3}f_1 + 5t_5\partial_{t_5}f_1 + \dots) + \partial_{t_3}f_1 = 0, \tag{6.12}$$

$$3(f_0 + 3t_3\partial_{t_3}f_0 + 5t_5\partial_{t_5}f_0 + \dots)^2(f_1 + 3t_3\partial_{t_3}f_1 + 5t_5\partial_{t_5}f_1 + \dots) + \partial_{t_5}f_1 = 0, \tag{6.13}$$

$$\dots$$

In principle one can hope these equations would specify a set of t_{2k-1} such that (6.12,6.13,...) are satisfied, which would ensure equivalence between the eigenstate and generalized Gibbs ensembles at the level of the expectation values of J_{2k} at first subleading order in $1/c$. It should be noted though that the equations (6.12,6.13,...) have no small parameter, because when written in terms of variables t_{2k-1} they are c-independent. Thus one would need to know full function $f_1(t_3, t_5, \dots)$ to find a possible solution. In this sense the problem of matching fugacities μ_{2k-1} to a primary state is non-perturbative, i.e. it requires knowledge of free energy at all orders in μ_{2k-1}, even for large c [37].

We will postopone this discussion and its physical implication to the next section. Here we will just give a simple argument that shows that a solution of (6.12) and more generally of (4.26) does not exist in general for finite but large c if we restrict ourself to a **finite** number of Q_{2k-1} charges in GGE ensemble.

Let us compare (6.9) with (6.10) for $\hat{Q}_3$ by calculating the difference between the two,

$$\ell^{-1}\left(\langle\hat{Q}_3\rangle_{GGE}-\langle E|\hat{Q}_3|E\rangle\right)=\ell^{-1}\left(\langle Q_3^l\rangle_{GGE}+\langle\tilde{Q}_3\rangle_{GGE}\right)-\left(\frac{E}{\ell^2}\right)^2. \tag{6.14}$$

Using explicit expression for $\hat{Q}_3$ (4.13) we find that in the thermodynamic limit the expectation value of $\hat{Q}_3$ is given by the saddle point,

$$\ell^{-1}\langle\hat{Q}_3\rangle_{GGE}=\left(\frac{E^*}{\ell^2}\right)^2+O(1/\ell). \tag{6.15}$$

The saddle point value $E^*=\Delta^*+n^*$ is equal to the energy E of state $|E\rangle$ due to (6.11). The values of Δ^*, n^* should be determined from the full effective action $\mathcal{L}+\tilde{\mathcal{L}}$ (4.18), (4.32). Both Δ^* and n^* scale with the system size as $\sim\ell^2$. Furthermore, up to $1/c$ corrections $\Delta^*\approx E^*=E$ and therefore Δ^*/ℓ^2 is a positive number in the thermodynamic limit. Using (6.14) and (6.15) we find the discrepancy between the eigenstate and GGE expectation values to be

$$\ell^{-1}\left(\langle\hat{Q}_3\rangle_{GGE}-\langle E|\hat{Q}_3|E\rangle\right)=\ell^{-1}\langle\tilde{Q}_3\rangle_{GGE}= \tag{6.16}$$
$$\frac{e^{-\tilde{\mathcal{L}}(\Delta^*,n^*)}}{\ell\,P(n^*)}\sum_{\{m\}=n^*}\langle m_i,\Delta^*|\tilde{Q}_3e^{-\mu_3\tilde{Q}_3-\mu_5\tilde{Q}_5-\dots}|m_i,\Delta^*\rangle.$$

So far we are interested in the leading in c behavior of (6.16) we can substitute $\tilde{Q}_3$ by a lower-triangular matrix $c\,\tilde{Q}_3^c+\Delta^*\tilde{Q}_3^\Delta$, $\tilde{Q}_5$ can be substituted by the lower-triangular $c^2\tilde{Q}_5^{cc}+c\,\Delta^*\tilde{Q}_5^{c\Delta}+(\Delta^*)^2\tilde{Q}_3^{\Delta\Delta}$, and so on. Then average of $\Delta^*\tilde{Q}_3^\Delta$ is simply equal to $4\Delta^*n^*/\ell^3$, while the average of $c\,\tilde{Q}_3^c$ can be rewritten as an average over Young tableaux,

$$\langle\tilde{Q}_3^c\rangle_{GGE}=\frac{e^{-\tilde{\mathcal{L}}(\Delta^*,n^*)}}{6\,\ell^3P(n^*)}\sum_{\{m\}=n^*}\left(\sum_i m_i^3\right)e^{-\frac{\mu_3}{\ell^3}\left(\frac{c}{6}\sum_i m_i^3+4\Delta^*n^*\right)-\frac{\mu_5}{\ell^5}\left(\frac{c^2}{72}\sum_i m_i^5+\dots\right)-\dots}+O(c^0).$$

In the thermodynamic limit this is equal to $a\,(n^*)^2/\ell^3$, where a is some non-negative function of $\mu_3,\mu_5,\dots$. When all $\mu_{2k-1}=0$, it reduces to $a_0=2/5$, (4.39). Thus we finally have

$$\ell^{-1}\left(\langle\hat{Q}_3\rangle_{GGE}-\langle E|\hat{Q}_3|E\rangle\right)=\frac{a\,(n^*)^2+4\Delta^*n^*}{\ell^4}+O(c^0). \tag{6.17}$$

The important point here is that the discrepancy (7.42) can be zero if and only if $n^* = 0$. The value of n^* can be interpreted as the effective level of Virasoro descendants which give main contribution to the partition function. It is a priory expected that the discrepancy between the primary state and the GGE would vanish if $n^* = 0$, i.e. if primaries dominate the partition sum. The non-trivial result here is that (7.42) vanishes only if $n^* = 0$.

It is easy to see that primaries do not dominate the partition sum because of the factor $P(n)$ accounting for the exponential growth of the number of descendants with n. Using (4.18), (4.41), (4.42), (4.53) the effective action for n can be rewritten in terms of the variable $\mathrm{n} = n/\ell^2$ which remains finite in the thermodynamic limit,

$$\mathcal{L}_{\text{eff}}(\mathrm{n}) = \ell\left(\pi\sqrt{\frac{2\mathrm{n}}{3}} - L\right), \tag{6.18}$$
$$L = (\beta + 6\mu_3\Delta^*/\ell^2 + \dots)\mathrm{n} + \mu_3(1 + ca_0 + b_0 + \dots)\mathrm{n}^2 + O(\mathrm{n}^{5/2}).$$

Since $L(\mathrm{n})$ admits an expansion in powers of n starting from one, $\mathrm{n} = 0$ can not be a solution of $\partial\mathcal{L}_{\text{eff}}/\partial\mathrm{n} = 0$ and the maximum of $\mathcal{L}_{\text{eff}}$ is achieved at positive $\mathrm{n} = n^*/\ell^2$. From here it follows that for large c both $\langle\hat{Q}_3\rangle_{GGE}$ and $\langle E|\hat{Q}_3|E\rangle$ scale as c^2, while their difference is non-zero and scales as c. This proves (6.9) and (6.10) for $\hat{Q}_3$ are always different for large but finite c for any values of μ_{2k-1}. A similar argument would also apply to $\hat{Q}_5$ and higher charges.

This argument shows that elimination of discrepancy between (6.9) and (6.10) is not possible on pertubative level for all sufficiently excited states. We will discuss validity Generalized ETH and its nuances in the next section.

7 Generalized Eigenstate Thermalization Hypothesis

In this section we formulate Generalized ETH for 2d CFTs and analytically verify it. In the simplest form Eigenstate Thermalization Hypothesis requires the expectation value of some appropriate (often taken to be local) observable $\mathcal{O}$ in a many-body eigenstate $|E_i\rangle$ to be a smooth function of energy,

$$\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(E_i). \tag{7.1}$$

Qualitatively, eq.(7.1) postulates that energy is the only thermodynamically relevant quantity, which completely specifies local properties of an eigenstate. The condition (7.1) may apply to all or most eigenstates, in which case it is referred as strong or weak ETH. The eigenstate thermalization ensures equivalence between the expectation value in the eigen-ensemble, $f_{\mathcal{O}}(E_i)$, and thermal expectation value of $\mathcal{O}$ in the Gibbs

ensemble, $f_{\mathcal{O}}(E_i) = \mathrm{Tr}\big(e^{-\beta H}\mathcal{O}\big)/Z$, where the effective temperature β is fixed through the energy balance relation, $E_i = \mathrm{Tr}\big(e^{-\beta H}\mathcal{O}\big)/Z$ [45].

When the system is integrable, with an extensive number of conserved charges $\hat{Q}_i$, ETH does not apply, as we have seen. Accordingly emerging equilibrium can be different from the Gibbs state. In this case it is believed that the equilibrium can be described by the Generalized Gibbs Ensemble (GGE), which includes an infinite tower of conserved charges [46]. Validity of the GGE has been related to the generalized eigenstate thermalization [6–8], which generalizes (7.1) to include an infinite number of conserved quantities,

$$\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(Q_k(E_i)). \tag{7.2}$$

Here $|E_i\rangle$ is a mutual eigenstate of the Hamiltonian and charges $\hat{Q}_k$, $Q_k(E_i)$ are the eigenvalues of $\hat{Q}_k$ associated with $|E_i\rangle$, and function $f_{\mathcal{O}}$ is assumed to be a smooth function of all of its arguments. Similarly to (7.1), at the qualitative level, (7.2) postulates that charges $\hat{Q}_k$ form a complete set of thermodynamically relevant quantities which fully specify local properties of an eigenstate. Provided (7.2) applies to most states, it ensures equivalence between the generalized microcanonical ensemble and GGE, establishing validity for the latter to describe emerging equilibrium e.g. following a quantum quench [6].

At the same time emergence of thermal equilibrium is not universal. Previously it was shown that the values of different local quantities measured in the individual energy eigenstates differ from their thermal counterparts, unless central charge is taken to be infinite [36, 39–42, 47–50].

By analogy with the integrable lattice models it is natural to expect that locally equilibrium states can be described in terms of the GGE, which includes all local qKdV charges. Indeed, emergence of exactly such qKdV GGE was analytically shown for a special family of so-called Cardy-Calabrese initial states [4, 51].

In the context of integrable systems the question which quantities should be included in the GGE is far from being trivial. Early studies in the context of XXZ and Lieb-Liniger models have shown that a full set of extensive local charges does not specify local properties of eigenstates, signaling failure of generalized ETH [52, 53]. These works raised an important question of the validity of the GGE to describe an emerging equilibrium following a quantum quench [54–56]. A resolution comes from the fact that besides local conserved quantities these models give rise to quasi-local conserved charges [5]. Taking them into account restores validity of the GGE [57]. Following studies in the context of integrable field theoretic models, both free and interacting ones, have decisively established that adding quasi-local charges is necessary to accurately describe the after-quench equilibrium state [58–63]. These findings raise an important

question emphasized in [51] if the set of local qKdV charges is generally sufficient to describe equilibrium in large c 2d CFTs, or it should be extended by non-local or perhaps some new local charges [64]. In this section we show that at large c, 2d CFTs satisfy generalized eigenstate thermalization (7.2) with the local qKdV charges forming a complete set and ambiguously specifying local properties of the eigenstates.

Our consideration below applies to all large central charge theories with $c \gg 1$. In the case theory admits a gravity dual, many aspects can be interpreted holographically. Below we provide holographic interpretation when available without requiring the holographic description to exist.

7.1 Setup

Two-dimensional conformal field theories admit a split into non-interacting sectors of left and right movers. For simplicity we only discuss one sector explicitly, while all results automatically extend to the full theory. As previously, we consider 2d CFT on a circle of the circumference ℓ in a mutual eigenstate of all charges Q_{2k-1},

$$|E\rangle = |\{m_i\}, \Delta\rangle, \quad E = Q_1 = (\Delta + \sum m_i)/\ell, \tag{7.3}$$

labeled by the primary state Δ and the set of integers $\{m_i\}$ [65]. The set $\{m_i\}$ we will parametrize using free boson representation where an integer n_k for $k = 1, 2, \ldots$ In the thermodynamic limit $\ell \to \infty$ "energy" Q_1 and all other qKdV charges are assumed to scale with the system size to yield finite charge densities $q_{2r-1} = Q_{2r-1}/\ell$. In terms of Δ, n_k this implies scaling

$$\Delta \sim \ell^2, \quad \sum_k n_k\, k^{2r-1} \sim \ell^{2r}. \tag{7.4}$$

In what follows we restrict the discussion to the eigenstates (7.3) with the density charges $q_{2r-1} = \langle E|\hat{Q}_{2r-1}|E\rangle/\ell$ which additionally satisfy

$$\frac{q_{2r-1}}{q_1^r} = 1 + O(1/c). \tag{7.5}$$

Here and below the CFT central charge c is assumed to be large. Holographically, this regime corresponds to a quasi-classical black hole in AdS_3, where one in the RHS of (7.5) corresponds to classical gravity, while $O(1/c)$ term is due to quantum corrections [29, 43, 44, 65]. In terms of Δ, n_k an exponential majority of states in the generalized microcanonical ensemble specified by q_{2k-1} subject to (7.5) will satisfy

$$\frac{\sum_k n_k\, k^{2r-1}}{\Delta^r} = O(1/c^r). \tag{7.6}$$

Typicality of (7.6) is justified in the following subsection. In fact (7.6) may apply to all states in the generalized microcanonical ensemble (7.5), yielding strong version of the generalized ETH in 2d CFTs. To verify that one would need to know full spectrum of qKdV charges, going beyond currently known leading $1/c$ expansion.

In the regime of "quasi-classical gravity" (7.6), $c \gg 1$, expectation values of qKdV charges are determined by (3.40),

$$\ell^2 q_1 = \Delta + \sum_k n_k k, \tag{7.7}$$

$$\ell^4 q_3 = \Delta^2 + \sum_k n_k \left(6\Delta k + \frac{c\, k^3}{6}\right) + O(c^0), \tag{7.8}$$

$$\dots$$

$$\ell^{2r} q_{2r-1} = \Delta^r + \sum_k n_k\, p_{2r-1}(c, \Delta, k) + O(c^{r-2}), \tag{7.9}$$

where $p_{2r-1}(c, \Delta, k)$ are some known polynomials of degree $2r-1$ which include only odd powers of k.

7.2 Explicit GETH calculation

Because of translational invariance the expectation value of a full derivative $\mathcal{O} = \partial \mathcal{O}'$ in energy eigenstate will vanish. Hence it suffices to consider expectation values $\langle E|\mathcal{O}|E\rangle$ only when $\mathcal{O}$ is a quasi-primary operator. Below we consider the case when $\mathcal{O}$ belongs to the vacuum family, i.e. it is a Virasoro descendant of the identity. To streamline the notations we introduce $\langle \mathcal{O}\rangle \equiv \langle E|\mathcal{O}|E\rangle$. It is convenient to parametrize $\mathcal{O}$ by its dimension (level). At the levels 2 and 4 there are unique quasi-primaries in the vacuum family,

$$\mathcal{O}_2 = T, \qquad \mathcal{O}_4 = T^2 - \frac{3}{10}\partial^2 T. \tag{7.10}$$

Thus expectation values of $\mathcal{O}_{2,4}$ are identically equal to charge densities q_1, q_3 [37]. At the level 6 there are two quasi-primaries (we always choose quasi-primaries in the basis which diagonalizes Zamolodchikov metric)

$$\mathcal{O}_6^{(1)} = T^3 - \frac{9}{10}(T\partial^2 T) + \frac{4}{35}\partial^4 T + \frac{93}{70c+29}\mathcal{O}_6^{(2)}, \tag{7.11}$$

$$\mathcal{O}_6^{(2)} = (\partial T \partial T) - \frac{4}{5}(T\partial^2 T) + \frac{23}{210}\partial^4 T. \tag{7.12}$$

The expectation value of the combination $\mathcal{O}_6^{(1)} + \frac{5}{9}\frac{c}{12}\mathcal{O}_6^{(2)}$ is identically equal to q_5. Similarly to (7.7-7.9), at leading order the expectation value of $\mathcal{O}_6^{(2)}$ has the form of a

polynomial in Δ and odd powers of k,

$$\langle \mathcal{O}_6^{(2)} \rangle = \frac{9}{5} \sum_k n_k \left(\frac{c}{6} k^5 + 4\Delta k^3 \right) + O(c^0). \tag{7.13}$$

It is possible to use (7.7-7.9) to express any term of the form $\sum_k n_k \, k^{2r-1}$ via q_{2j-1}, $j \leq r$, but a priori the result would also depend on Δ. Thus, at leading order in $1/c$, expectation values of $\mathcal{O}_6^{(i)}$ are some functions of Δ and q_{2r-1}. Remarkably, because of the non-trivial cancellations the final result is Δ-independent, and can be expressed solely in terms of q_{2r-1}. To simplify the answer we introduce dimensionless ratio $\mathrm{q}_{2k-1} = q_{2k-1}/q_1^k$ such that $\delta \mathrm{q}_{2k-1} \equiv \mathrm{q}_{2k-1} - 1$ is of order $1/c$. Then $\mathcal{O}_6^{(i)}$ measured in units of energy density q_1 is given by

$$q_1^{-3} \langle \mathcal{O}_6^{(1)} \rangle = 1 + 3\, \delta \mathrm{q}_3 + O(1/c^2), \tag{7.14}$$

$$q_1^{-3} \langle \mathcal{O}_6^{(2)} \rangle = \frac{9}{5} \frac{12}{c} \left(\delta \mathrm{q}_5 - 3\, \delta \mathrm{q}_3 \right) + O(1/c^3). \tag{7.15}$$

As we see different quasi-primaries have different scaling with c. Our calculation applies to leading $1/c$ behavior of each quasi-primary, except for a special one, which includes maximal power of T without derivatives. The expectation value of that quasi-primary starts with $O(c^0)$ and our result applies to the first two terms in $1/c$ expansion.

The possibility to express eigenstate expectation value $\langle \mathcal{O} \rangle$ as a polynomial in q_{2j-1} extends to all higher levels. For an operator of dimension $2r$ the answer only depends on q_{2j-1} for $j \leq r$. We write down explicit expressions for all operators up to level 10 in terms of q_{2j-1} in the next section. Our results establish generalized eigenstate thermalization for vacuum block observables in large c CFTs.

That expectation value $\langle \mathcal{O} \rangle$ of an operator of dimension $2r$ only includes qKdV charges q_{2j-1} up to the same dimension $j \leq r$ can be interpreted as a manifestation of locality. It is analogous to the observation in the context of integrable lattice models that to describe equilibrium state locally, at the length scales not exceeding some distance a, it is only necessary to include local and quasi-local charges in the GGE with the support within a [66, 67].

7.3 Expectation value of quasi-primaries in eigenstates

In this section we list the explicit expressions for the eigenstate expectation values of all quasi-primaries up to level ten in terms of qKdV charges.

7.3.1 Level 6

There is are two quasi-primaries

$$\mathcal{O}_6^{(1)} = T^3 - \frac{9}{10}(T\partial^2 T) + \frac{4}{35}\partial^4 T + \frac{93}{70c+29}\mathcal{O}_6^{(2)}, \tag{7.16}$$

$$\mathcal{O}_6^{(2)} = (\partial T \partial T) - \frac{4}{5}(T\partial^2 T) + \frac{23}{210}\partial^4 T. \tag{7.17}$$

In the limit (7.6) they can be simplified to

$$\mathcal{O}_6^{(1)} = T^3 + O(1/c), \tag{7.18}$$

$$\mathcal{O}_6^{(2)} = \frac{9}{5}(\partial T \partial T) + O(1/c). \tag{7.19}$$

In units of the energy density their expectation values are

$$q_1^{-3}\langle \mathcal{O}_6^{(1)} \rangle = 1 + 3\,\delta \mathrm{q}_3 + O(1/c^2), \tag{7.20}$$

$$q_1^{-3}\langle \mathcal{O}_6^{(2)} \rangle = \frac{9}{5}\frac{12}{c}\,(\delta \mathrm{q}_5 - 3\,\delta \mathrm{q}_3) + O(1/c^3). \tag{7.21}$$

7.3.2 Level 8

There are three quasi-primaries at level 8,

$$\mathcal{O}_8^{(1)} = T^4 + O(1/c), \tag{7.22}$$

$$\mathcal{O}_8^{(2)} = \frac{9}{5}(T(\partial T \partial T)) + O(1/c), \tag{7.23}$$

$$\mathcal{O}_8^{(3)} = \frac{143}{63}(\partial^2 T \partial^2 T) + O(1/c). \tag{7.24}$$

In the units of energy density at leading order they are

$$q_1^{-4}\langle \mathcal{O}_8^{(1)} \rangle = 1 + 6\,\delta \mathrm{q}_3 + O(1/c^2), \tag{7.25}$$

$$q_1^{-4}\langle \mathcal{O}_8^{(2)} \rangle = \frac{9}{5}\frac{12}{c}\,(\delta \mathrm{q}_5 - 3\,\delta \mathrm{q}_3) + O(1/c^3), \tag{7.26}$$

$$q_1^{-4}\langle \mathcal{O}_8^{(3)} \rangle = \frac{143}{63}\frac{180}{c^2}\,(\delta \mathrm{q}_7 - 4\,\delta \mathrm{q}_5 + 6\,\delta \mathrm{q}_3) + O(1/c^4).$$

7.3.3 Level 9

There are no quasi-primaries of odd dimension smaller than nine. At level nine there is a unique quasi-primary $\mathcal{O}_9$, which has zero expectation value, as well as all higher odd-dimensional quasi-primaries, due to parity.

7.3.4 Level 10

There are four quasi-primaries at level 8. In the limit (7.6) up to some additional factors they are

$$\mathcal{O}^{(1)}_{10} = T^5 + O(1/c), \tag{7.27}$$
$$\mathcal{O}^{(2)}_{10} = (T(T(\partial T \partial T))) + O(1/c), \tag{7.28}$$
$$\mathcal{O}^{(3)}_{10} = (T(\partial^2 T \partial^2 T)) + O(1/c), \tag{7.29}$$
$$\mathcal{O}^{(4)}_{10} = (\partial^3 T \partial^3 T) + O(1/c). \tag{7.30}$$

In terms of energy density their expectation values are

$$q_1^{-5}\langle \mathcal{O}^{(1)}_{10} \rangle = 1 + 10\,\delta \mathrm{q}_3 + O(1/c^2), \tag{7.31}$$
$$q_1^{-5}\langle \mathcal{O}^{(2)}_{10} \rangle = \frac{1}{c}\,(\delta \mathrm{q}_5 - 3\,\delta \mathrm{q}_3) + O(1/c^3), \tag{7.32}$$
$$q_1^{-5}\langle \mathcal{O}^{(3)}_{10} \rangle = \frac{180}{c^2}\,(\delta \mathrm{q}_7 - 4\,\delta \mathrm{q}_5 + 6\,\delta \mathrm{q}_3) + O(1/c^4), \tag{7.33}$$
$$q_1^{-5}\langle \mathcal{O}^{(4)}_{10} \rangle = \frac{3024}{c^3}\,(\delta \mathrm{q}_9 - 5\,\delta \mathrm{q}_7 + 10\,\delta \mathrm{q}_5 - 10\,\delta \mathrm{q}_3) + O(1/c^5).$$

7.4 Typicality of "quasi-classical gravity" regime

In this section we discussed mutual eigenstates of qKdV hierarchy which additionally satisfy the set of condition (7.6). We refer to this set of conditions as the "quasi-classical gravity" regime without actually requiring the underlying theory to be holographic. Below we show that among the states from the microcanonical ensemble satisfying (7.5) an exponential majority of states satisfy (7.6). Furthermore, most CFT eigenstates satisfy (7.5).

We start with the microcanonical ensemble specified by the mean energy density $q_1 = Q_1/\ell$, $\ell \to \infty$. Up to the terms negligible in the thermodynamic limit "energy" is a sum of the primary dimension Δ and the descendant level n,

$$Q_1 = \frac{\Delta + n}{\ell}, \qquad n = \sum_k n_k\, k. \tag{7.34}$$

For large $\Delta \to \infty$ density of primaries is given by the Cardy formula $P_\Delta \propto e^{\pi\sqrt{2c\Delta/3}}$ while the density of descendants is controlled by the number of integer partitions, $P_n \propto e^{\pi\sqrt{2n/3}}$. Thus, typical state has

$$\frac{\Delta}{\ell^2} \sim q_1 \left(1 - \frac{1}{c}\right), \quad \frac{n}{\ell^2} \sim \frac{q_1}{c}, \tag{7.35}$$

with the deviations being exponentially suppressed as we have seen in section 4.

Next, we impose an additional condition from (7.5), $q_3/q_1^2 - 1 = O(1/c)$. Taking into account that in the thermodynamic limit $\ell \to \infty$

$$q_3 = \frac{Q_3}{\ell} = \frac{\Delta^2 + 6\Delta n + \frac{c}{6}\sum_k n_k\, k^3}{\ell^4}, \tag{7.36}$$

and using (7.35), we find

$$\frac{\sum_k n_k\, k^3}{\Delta^2} \sim O(1/c^2), \tag{7.37}$$

in full agreement with (7.6).

As a next step we impose $q_5/q_1^3 - 1 = O(1/c)$ and using

$$q_5 = \frac{Q_5}{\ell} = \frac{\Delta^3 + 15\Delta^2 n + \frac{5c}{6}\Delta\sum_k n_k\, k^3 + \frac{c^2}{72}\sum_k n_k\, k^5}{\ell^3},$$

as well as (7.35,7.37), we find

$$\frac{\sum_k n_k\, k^5}{\Delta^3} \sim O(1/c^3), \tag{7.38}$$

again, in full agreement with (7.6). Continuing this logic, we find that an exponential majority of states in the generalized microcanonical ensemble (7.5) satisfy (7.6). The inverse is obviously correct as well: all states satisfying (7.6) automatically satisfy (7.5).

Current theoretical limitations do not allow us to analyze qKdV eigenstates which do not satisfy (7.5). Nevertheless most states in the CFT spectrum automatically belong to the generalized microcanonical shell (7.5). To see that, we return to the condition (7.35), which is typical for all states withing the microcanonical shell specified by the energy density q_1. In section 4.3.3 we have calculated mean value of $\sum_k n_k\, k^3$ keeping n fixed, which turned out to be equal to $2/5n^2$, while the standard deviation is $1/\sqrt{n} \propto 1/\ell$ suppressed. Thus, typical states with the specified energy density automatically satisfy (7.37). More generally, using the computational technique from the section 4.3.3 one can see that the mean value of

$$\frac{\sum_k n_k\, k^{2p-1}}{n^p}, \qquad p \geq 1, \tag{7.39}$$

is of order one in the limit $\ell \to \infty$, while the standard deviation is always $1/\sqrt{n} \propto 1/\ell$ suppressed. Using (7.35) one readily sees that most qKdV eigenstates satisfy (7.5).

7.5 Beyond $1/c$ and holography

Our results capture only leading order in $1/c$. It remains an open question to find exact expressions for the eigenstate expectation values in terms of the qKdV charges, as was recently done for a particular family of integrable spin models [68, 69].

Generalized eigenstate thermalization implies validity of the qKdV Generalized Gibbs Ensemble

$$\rho = \exp\left\{-\sum_k \mu_{2k-1} Q_{2k-1}\right\}/Z, \quad \mu_1 \equiv \beta, \tag{7.40}$$

to describe local properties of individual energy eigenstates, provided chemical potentials μ_{2k-1} are tuned to match values of the eigenstate charges

$$\ell\, q_{2k-1} = \langle E_i|Q_{2k-1}|E_i\rangle = \mathrm{Tr}(\rho\, Q_{2k-1}). \tag{7.41}$$

Provided q_{2k-1} a chosen to represent charge densities of some non-equilibrium initial state $|\Psi\rangle$, a standard argument would consequently equate the GGE expectation values of local operators with those in the diagonal ensemble of $|\Psi\rangle$, written in the eigenbasis (7.3). In most cases the latter would be equal to the expectation values in state $|\Psi\rangle$ upon equilibration. It should be noted though that left and right Hamiltonians $Q_1, \bar{Q}_1$ are highly degenerate, and therefore validity of the diagonal ensemble to describe local physics upon equilibration may be violated.

It remains an open question to establish existence of μ_{2k-1} which would solve (7.41) for any given set of q_{2k-1}. Using explicit form of the generalized partition function in the large c limit (4.90) we can find, up to $O(1/c^2)$ corrections,

$$\begin{aligned}
&\delta \mathrm{q}_{2k-1} = \frac{q_{2k-1}}{q_1^k} - 1 = \\
&\frac{24k}{c}\int_0^\infty \frac{d\kappa\kappa\left[(2k-1)_2F_1(1,1-k,3/2,-\kappa^2)-1\right]}{e^{2\pi\kappa\gamma}-1}, \\
&\gamma = \sum_{j=1}^{\infty} \tilde{\mu}_{2j-1} j(2j-1)\sigma^{j-1/2}{}_2F_1(1,1-j,3/2,-\kappa^2),
\end{aligned} \tag{7.42}$$

where $\tilde{\mu}_{2k-1} = \frac{\sqrt{6}}{\pi} c^{k-1}\mu_{2k-1}$ and $\sigma(\tilde{\mu}_{2k-1})$ is positive and satisfies

$$\sum_{k=1} k\, \tilde{\mu}_{2k-1}\, \tilde{\sigma}^{k-1/2} = 1. \tag{7.43}$$

From here it follows that when all chemical potentials are positive q_{2k-1} satisfy an infinite series of inequalities (see next subsection 7.5.1)

$$\begin{aligned}
&\frac{q_3}{q_1^2} - 1 \le \frac{22}{5c} + O(1/c^2), \quad \frac{q_5}{q_1^3} - 1 \le \frac{302}{21c} + O(1/c^2), \\
&\ldots
\end{aligned} \tag{7.44}$$

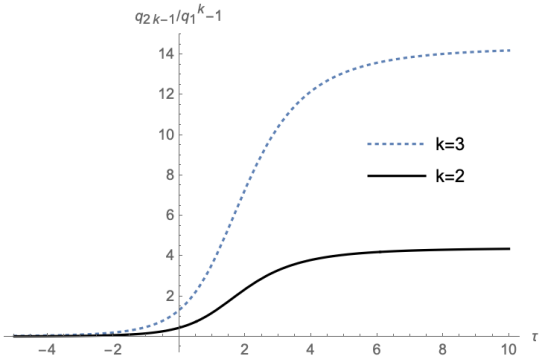

Figure 2. Plot of $q_{2k-1}/q_1^k - 1$ in the units of $1/c$ as a function of $\tau = \beta(\pi^2/(6c\mu_3))^{1/3}$ for $k = 2, 3$. It approaches zero as $|\tau|^{-3}$ for all k when $\tau \to -\infty$. The opposite limit $\tau \to \infty$ corresponds to the Gibbs ensemble, $q_1 \sim \beta^{-1}$, $\mu_3 \to 0$, and $c(q_{2k-1}/q_1^k - 1)$ for $k = 2, 3$ approach 22/5 and 302/21 correspondingly.

Thus GGE emerging after equilibration of some general initial state will have to include negative chemical potentials, unless all inequalities (7.44) are satisfied.

To match GGE to a primary state all qKdV densities should be related to each other via $q_{2k-1} = q_1^k$ [37]. This is only possible if the integral in (7.42) vanishes, which requires γ to be infinite. This is consistent with the observation of [65] that an ensemble with any finite number of non-zero μ_{2k-1} can not describe primary states. This is because in full generality $q_{2k-1} \geq q_1^k$ and hence primary states are at the boundary of the phase space of q_{2k-1}'s. It is nevertheless possible to describe them in the limit, via a GGE with at lest some coefficients approaching infinity. The simplest scenario is to consider $\mu_3 > 0$ and arbitrary $\beta \equiv \mu_1$, while all other chemical potentials are identically zero. Then in the limit $\tau = \beta(6/\pi^2 c\mu_3)^{1/3} \to -\infty$, for all k, $q_{2k-1}/q_1^k - 1$ will vanishes as $\sim |\tau|^{-3}$, as is shown for $k = 2, 3$ in Fig. 2.

With just two chemical potentials β, μ_3 being non-zero the values of $q_{2k-1}/q_1^k - 1$ is confined to be between zero and their thermal (Gibbs ensemble) values. This constraint is removed already after turning on one more additional chemical potential. For example by taking $\beta, \mu_5 > 0$ and $\mu_3 < 0$ one can fine-tune function γ to become arbitrarily small for some positive value of κ, leading to the divergence of the integral in (7.42) and violating quasi-classical regime (7.5).

From the holographic point of view equilibration in field theory is associated with the formation of a black hole in AdS_3, a background dual to the GGE (7.40). Conserved qKdV charges correspond to the black hole soft hair, which are only visible at quantum

level. At the level of classical gravity $c \to \infty$ all qKdV charges are related, $q_{2k-1} = q_1^k$. Accordingly there is a unique classical BTZ black hole family of solutions parametrized by $q_1, \bar{q}_1$ [43, 44]. It is an important question to understand the regime $q_{2k-1} \neq q_1^k$ holographically, by including quantum gravity corrections into consideration. This, in particular, should provide holographic interpretation to negative temperature and other chemical potentials, which will necessarily appear starting from a general initial state.

In this section we have only considered local probes $\mathcal{O}$ from the vacuum block. In case $\mathcal{O}$ is a non-trivial Virasoro primary, or its descendant, it will have zero expectation value in the GGE (7.40) for any values of μ_{2k-1}. This is because in the thermodynamic limit $\ell \to \infty$ geometry degenerates into a cylinder, which is conformally flat. Thus, to satisfy any version of eigenstate thermalization the eigenstate expectation value $\langle E|\mathcal{O}|E\rangle$ must simply vanish. In terms of the CFT data, this means most or all heavy-heavy-light Operator Product Expansion coefficients must approach zero when the dimension of heavy operators grows to infinity. If that is the case, generalized eigenstate thermalization will be trivially satisfied. It remains an outstanding problem to establish if large central charge chaotic CFT, in particular those with gravity duals, exhibit this behavior.

7.5.1 GGE with positive chemical potentials

For any positive integer j hypergeometric function ${}_2F_1(1, 1-j, 3/2, -\kappa^2)$ is polynomial in κ^2 with non-negative coefficients which starts with one,

$$ {}_2F_1(1, 1-j, 3/2, -\kappa^2) = 1 + \frac{2}{3}(j-1)\kappa^2 + \dots \tag{7.45}$$

Hence it is a monotonically increasing function of κ which satisfies ${}_2F_1(1, 1-j, 3/2, -\kappa^2) \geq 1$. From here it follows that when all chemical potentials are non-negative, function γ defined in the equation (7.42) satisfies

$$\gamma \geq \sum_{j=1}^{\infty} \tilde{\mu}_{2j-1} j(2k-1)\sigma^{j-1/2} \geq \sum_{j=1}^{\infty} \tilde{\mu}_{2j-1} j \sigma^{j-1/2} = 1.$$

Thus at leading order in $1/c$, $q_{2k-1}/q_1^k - 1$ is bounded from above by its value in the Gibbs ensemble,

$$\begin{aligned} &\delta \mathrm{q}_{2k-1} \leq \\ &\frac{24k}{c} \int_0^{\infty} \frac{d\kappa \kappa \left[(2k-1)\,{}_2F_1(1, 1-k, 3/2, -\kappa^2) - 1\right]}{e^{2\pi\kappa} - 1} = \\ &\frac{k}{c} \left(\sum_{p=0}^{k-1} \frac{6(2k-1)\Gamma(k)\Gamma(1/2)}{\Gamma(p+3/2)\Gamma(k-p)} (-1)^{p+1} \zeta(-1-2p) - 1 \right). \end{aligned} \tag{7.46}$$

This yields 22/5 for $k=2$, 302/11 for $k=3$, 2428/75 for $k=4$, and so on.

7.5.2 GGE with two non-zero chemical potentials

To gain better intuition it is instructive to consider the generalized ensemble which includes only two charges, the conventional Hamiltonian of CFT $H \equiv Q_1$ and Q_3,

$$\rho = \exp\left(-\beta H - \mu_3 Q_3\right)/Z. \tag{7.47}$$

To assure convergence we must require $\mu_3 > 0$ while β can be arbitrary. It is convenient to parametrize β, μ_3 in terms of

$$\tau = \beta\left(\frac{6}{\pi^2 c \mu_3}\right)^{1/3}, \tag{7.48}$$

and energy density $q_1 = -\ell^{-1}\frac{\partial \ln Z}{\partial \beta}$, such that

$$\begin{aligned}
\beta &= q_1^{-1/2}\left(\frac{c\pi^2}{6}\right)^{1/2}\frac{\tau\left(\sqrt[3]{\tau^3+3\left(\sqrt{6\tau^3+81}+9\right)}-\tau\right)}{\sqrt{6}\sqrt[6]{\tau^3+3\left(\sqrt{6\tau^3+81}+9\right)}},\\
\mu_3 &= q_1^{-3/2}\left(\frac{c\pi^2}{6}\right)^{1/2}\frac{\left(\sqrt[3]{\tau^3+3\left(\sqrt{6\tau^3+81}+9\right)}-\tau\right)^3}{6\sqrt{6}\sqrt{\tau^3+3\left(\sqrt{6\tau^3+81}+9\right)}}.
\end{aligned}$$

Then $\delta \mathrm{q}_{2k-1}$ only depends on τ,

$$\begin{aligned}
\gamma &= 1+\frac{2^{1/2}\left(\kappa^2+1\right)\left(\sqrt[3]{\tau^3+3\left(\sqrt{6\tau^3+81}+9\right)}-\tau\right)^3}{3^{3/2}\sqrt{\tau^3+3\left(\sqrt{6\tau^3+81}+9\right)}},\\
\delta \mathrm{q}_{2k-1} &= \\
&\frac{24k}{c}\int_0^\infty \frac{d\kappa\,\kappa\left[(2k-1)_2F_1(1,1-k,3/2,-\kappa^2)-1\right]}{e^{2\pi\kappa\gamma}-1}.
\end{aligned} \tag{7.49}$$

When τ approaches minus infinity while q_1 is kept fixed,

$$\beta \sim -q_1^{-1/2}\left(\frac{c\pi^2}{6}\right)^{1/2}|\tau|^{3/2}2^{-1/2}, \tag{7.50}$$

$$\mu_3 \sim q_1^{-3/2}\left(\frac{c\pi^2}{6}\right)^{1/2}|\tau|^{3/2}2^{-3/2}, \tag{7.51}$$

and we find that $c\delta \mathrm{q}_{2k-1}$ approaches zero as $1/|\tau|^3$. We plot $\delta \mathrm{q}_{2k-1} \equiv q_1^{-k} q_{2k-1} - 1$ in the units of $1/c$ as a function of τ for $k=2,3$ in Fig. 2 in the main text.

8 Conclusions and Discussion

Here we just outline major results we obtained and mention some open problems.

Major mathematical component of this thesis is the spectrum of quantum KdV charges (3.39). We obtained this spectrum via a classical computation and careful consideration of quantum corrections. As we mentioned, ad hoc "quantization" (3.38) does not fully work and the data of "energies" of primary states is required to fully fix the quantum spectrum. However, it is not clear how to extend this "algorithm" to all orders in central charge c. It is may be enough to just know the "energies" of primary states at the relevant order in c in appropriate form, as we have seen at several first orders in $1/c$. Therefore, the question of establishing this "algorithm" in general remains open. Even more generally, one could ask whether it is possible to somehow "deform" the naive quantization scheme (3.38) in order to obtain full quantum spectrum without prior knowledge of primary states "energies".

Another mathematical result of this thesis is the partition function of GGE at leading relevant order in $1/c$. We only computed this partition function at first non-trivial order, where the sum consists of "free bosons" with nontrivial "masses". An obvius, though a tedious task would be to try to extend this computation further in $1/c$ given the known spectrum (3.39), where these bosons start to "interact". Another, perhaps, more attractive task is to find modular properties of this partition function. Standard partition function of 2d CFT is modular invariant. GGE partition function, at least naively, does not hold this property. This leaves a question how GGE partition function transforms under the action of modular group. See [34] and [70] for relevant discussion. Quantitative answer to this question would probably not only help to calculate GGE partition function non-perturbatively in $1/c$ but also would give some physical insights in the context of thermalization. Another possible implication is related to new stringent constraints on CFT spectrum via modular bootstrap.

On the physical side of the project, our main result is the methodology to analytically access properties of individual mutual eigenstates of the qKdV hierarchy in 2d CFTs. To the best of our knowledge, our work is the very first analytic proof of ETH in spatially-extended systems, with all previous works (with a very few exceptions) being numerical. However, some details remain unknown. Firstly, the set of states which oblige strong version of GETH is only established at large c limit. We expect that strong version of GETH does not necessary holds at finite central charge. Perhaps, some sort of a weaker version could be applied. In general, which 2d CFTs should be considered "chaotic" and which should not remains open. It is plausible that detailed analysis of thermalization of 2d CFTs beyond large c limit could shed some light on the "space of 2d CFTs" in that sense. Finally, holographic interpretation and implications

are far from complete and the implications of ETH in the context of Black Hole physics are still largely unknown. Holographically, GGE states correspond to BTZ black holes states. Logically one could develop a theory of generalized hydrodynamics describing long-wave dynamics of states locally deviating from the GGE. This description should be valid both field theory, and in the bulk, where it would describe the dynamics near a black hole background.

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