Thesis Changes Log

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PhD Program: Computational And Data Science And Engineering
Title of Thesis: On The Performance Of Quantum Approximate Optimization
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The thesis document includes the following changes in answer to the external review process:

1. Title changed as "On The Performance Of Quantum Approximate Optimization".
2. Page 3) Text rephrasing: This parameterized state, called ansatz, is then tuned iteratively via classical outer-loop optimization routines to minimize the expected value of some classical cost function.
3. Page 3) Text rephrasing: Secondly, we consider training QAOA circuits and show that the optimal circuit parameters feature a concentration effect, called parameter concentrations.
4. Page 3) Text rephrasing: We conjecture a logistic saturation behaviour for the circuit depth as a function of problem density to recover ϵ-tolerant approximations.We test our prediction against simulated data and find the model to be capable of describing the data within a 3σ confidence.
5. Page 8) New symbol added to List of Symbols : $\mathbb{N}, \mathbb{N}_0$ – Field of natural numbers and natural numbers inclusive of zero
6. Page 8) Text change: P– The class of problems that can be solved in polynomial time by a deterministic Turing machine.
7. Page 8) Text change: NP– The class of problems that can be verified in polynomial time by a deterministic Turing machine.
8. Page 14) Minor corrections in Statement 2:
   The optimal parameters for the fixed depth QAOA admit a concentration effect. Specifically, concentrations imply that, given a set of optimal parameters $\boldsymbol{\beta}^*_{\boldsymbol{n}}$ and $\boldsymbol{\gamma}^*_{\boldsymbol{n}}$, for n qubits, a set of optimal parameters for $n+1$ qubits, $\boldsymbol{\beta}^*_{\boldsymbol{n+1}}$ and $\boldsymbol{\gamma}^*_{\boldsymbol{n+1}}$ can be found such that:
   $$\left|\boldsymbol{\beta}^*_{\boldsymbol{n+1}} - \boldsymbol{\beta}^*_{\boldsymbol{n}}\right|^2 + \left|\boldsymbol{\gamma}^*_{\boldsymbol{n+1}} - \boldsymbol{\gamma}^*_{\boldsymbol{n}}\right|^2 \sim \text{poly}^{-1}(n).$$
   - Parameters concentrate for $p = 1, 2$ QAOA on variational state preparation with concentration scaling:
     $$\left|\boldsymbol{\beta}^*_{\boldsymbol{n+1}} - \boldsymbol{\beta}^*_{\boldsymbol{n}}\right|^2 + \left|\boldsymbol{\gamma}^*_{\boldsymbol{n+1}} - \boldsymbol{\gamma}^*_{\boldsymbol{n}}\right|^2 \sim n^{-4},$$
   - Parameters concentrate for $p \geq 3$ depth QAOA on variational state preparation for up to $p = 5$ and 17 qubits with observed scaling being the same as for $p = 1, 2$.
9. Page 19) Minor corrections in Definition 1.5:
   Given $\mathcal{H}$, the Hilbert space of n qubits and $A \in \mathcal{L}(\mathcal{H})$, a basis decomposition of A, in terms of Pauli strings, can be obtained as:
   $$A = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \bigotimes_{k=1}^{n} \sigma_k^{\alpha_k},$$

where $\boldsymbol{\alpha} \in \{0,1,2,3\}^{\times n}$ and $\sigma_k^{\alpha_k}$ represent the corresponding Pauli matrix according to the labeling $\sigma_k^0 = \mathbb{1}_k, \sigma_k^1 = X_k, \sigma_k^2 = Y_k$ and $\sigma_k^3 = Z_k$. The cardinality, $|A|$, is then defined as the number of non-zero coefficients $c_{\boldsymbol{\alpha}} \in \mathbb{C}$ in Eq. (1.6) and the locality given by the maximum number of non-identity terms taken over the Pauli strings, $\bigotimes_{k=1}^{n} \sigma_k^{\alpha_k}$.

10. Page 21) Minor corrections in Definition 1.8:
Given $A \in \mathrm{herm}_{\mathbb{C}}[2^n]$, the expected value given a unit norm state $|\psi\rangle \in \mathcal{H}$ is defined as:

$$\langle\psi| A |\psi\rangle = \sum_{k=1}^{2^n} \lambda_k |c_k|^2 ,$$

where λ_k are the eigenvalues of A and c_k are the coefficients of $|\psi\rangle$ in the eigenbasis of A.

11. Page 23) Equation Eq. (2.1) changed to: $f : \mathbb{B}^{\times n} \longrightarrow \mathbb{N}_0$.

12. Page 24) Equation Eq. (2.2) modified as:

$$\begin{aligned} \mathbb{B} &\longrightarrow \mathbb{C}^2 \\ \times &\longrightarrow \otimes. \end{aligned}$$

13. Page 28) Minor edits in the sentence: We map binary variables $x_j \longrightarrow \frac{1}{2}(\mathbb{1} + Z_j)$, where Z_j is the Pauli-Z matrix acting on the j^{th} qubit.

14. Page 33) Discussion added to the end of Chapter 2.

15. Page 39) Definition 3.4 corrected as:
Given a problem Hamiltonian $H > 0$, whose ground states we wish to approximate using p-depth QAOA, let $H_{\min}$ represent the ground state energy of H. The approximation ratio is then defined as:

$$r = \frac{H_{\min}}{\langle\psi(\boldsymbol{\gamma},\boldsymbol{\beta})| H |\psi(\boldsymbol{\gamma},\boldsymbol{\beta})\rangle}.$$

QAOA exactly recovers ground states whenever, $r = 1$.

16. Page 49) Discussion added to the end of Chapter 3.

17. Page 51) Definition 4.1 restated as:
Let Ω_p represent the variational state space accessible for a parameterized quantum circuit of depth p. Given some problem Hamiltonian to be minimized H, the circuit is underparameterized, whenever:

$$\min_{\psi\in\Omega} \langle\psi(\boldsymbol{\theta})| H |\psi(\boldsymbol{\theta})\rangle - \min_{\phi\in\mathcal{H}} \langle\phi| H |\phi\rangle > 0 \implies \Omega_p \subset \mathcal{H}.$$

18. Page 51) Definition 4.2 restated as:
Given some problem Hamiltonian $H(\alpha)$, on n-qubits and of density α, let Ω represent the variational state space accessible for a p-depth QAOA circuit:

$$\Omega = \bigcup_{\boldsymbol{\gamma},\boldsymbol{\beta}} \Big\{ |\psi(\boldsymbol{\gamma},\boldsymbol{\beta})\rangle \Big\},$$

where $|\psi(\boldsymbol{\gamma},\boldsymbol{\beta})\rangle$ as in Eq. (**??**). Reachability deficits are α induced underparameterization:

$$\min_{\psi\in\Omega} \langle\psi| H(\alpha) |\psi\rangle - \min_{\phi\in\mathcal{H}} \langle\phi| H(\alpha) |\phi\rangle > 0.$$

19. Page 70) Text rephrasing: Considering random instances of the satisfiability problem, we observe: instances with relatively low clause density require low depth QAOA circuits, whereas for high density instances, larger depth is required in order to approximate the minimum.

20. Page 72) Definition 5.2 restated as:
Let Γ represent a problem class, and let H_g represent the corresponding problem Hamiltonians for random instances $g \in \Gamma$. For QAOA circuits of fixed depth $p \in \mathbb{N}$, given $(\boldsymbol{\gamma}_n^*,\ \boldsymbol{\beta}_n^*) \in \arg\min_{\boldsymbol{\gamma},\boldsymbol{\beta}} \langle \psi(\boldsymbol{\gamma},\boldsymbol{\beta})|\, \mathrm{H}_g \,|\psi(\boldsymbol{\gamma},\boldsymbol{\beta})\rangle$. Parameters concentrate whenever:

$$\begin{aligned} &\forall\ (\boldsymbol{\beta}_n^*, \boldsymbol{\gamma}_n^*)\ \exists\ \left(\boldsymbol{\beta}_{n+1}^*, \boldsymbol{\gamma}_{n+1}^*\right) : \\ &\left|\boldsymbol{\beta}_{n+1}^* - \boldsymbol{\beta}_n^*\right|^2 + \left|\boldsymbol{\gamma}_{n+1}^* - \boldsymbol{\gamma}_n^*\right|^2 \sim \text{poly}^{-1}(n)\,. \end{aligned}$$

21. Page 79) Minor edits in Theorem 5.1:
Let $|t\rangle \in [\mathbb{C}_2]^{\otimes n}$ be an n qubit target state in the computational basis. For depth $p = 1, 2$, parameters concentrate as:

$$\left|\boldsymbol{\beta}_{n+1}^* - \boldsymbol{\beta}_n^*\right|^2 + \left|\boldsymbol{\gamma}_{n+1}^* - \boldsymbol{\gamma}_n^*\right|^2 = \mathcal{O}\left(\frac{1}{n^4}\right). \tag{1}$$

22. Page 88) Conjecture 6.1 restated as:
Critical depth $p^\star$, depends on clause density α,for MAX-2-SAT instances as:

$$p^*(\alpha) \approx \frac{p_{max}}{1 + e^{-\kappa(\alpha - \alpha_c)}}, \tag{2}$$

where α_c is the critical density, κ the logistic growth rate, and p_{max} the saturation value.

23. Page 92) Text rephrasing: The modern approach in quantum algorithms development is centered around the variational model of quantum computation, wherein a hybrid architecture, composed of a classical co-processor and a near term quantum device, is exploited to realize ground state quantum computing. In this framework, for classical combinatorial optimization tasks, QAOA is among the most studied with promising prospects in near-term applications.